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1. Introduction

The cyclic group, $\mathbb{Z}/2$, of order two plays a leading rôle in the theory of real vector bundles and manifolds. More precisely, it plays many parts, as abstract permutation group, as orthogonal group in dimension one, as Galois group of $\mathbb{C}$ over $\mathbb{R}$, now clearly distinguished, now merging one into another. The plot is not, by any means, fully revealed. We recount here but a few scenes in which $\mathbb{Z}/2$ figures.

The results largely occur in some form in the literature, many in the unpublished thesis [25] . Our purpose here is to present a concise account from the particular viewpoint of $\mathbb{Z}/2$-homotopy theory and without detailed proofs.

We begin with an extremely simple, but fundamental, result; as we shall see in §3, it lies very close to the theorem of Kahn and Priddy. Here and throughout the essay L will denote the real representation $\mathbb{R}$ of $\mathbb{Z}/2$ with the non-trivial action as multiplication by ± 1 .

Proposition (1.1) . Let ξ and ξ' be real vector bundles over a compact ENR X. Suppose that the sphere-bundles $S(\xi)$ and $S(\xi')$ are stably fibre-homotopy equivalent. Then $S(L\otimes\xi)$ and $S(L\otimes\xi')$ are $\mathbb{Z}/2$-equivariantly stably fibre-homotopy equivalent.

Notation. Any space may be considered as a $\mathbb{Z}/2$-space with the trivial action. We use the same symbol for the original

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space and the corresponding $\mathbb{Z}/2$-space. Thus, $S(\xi)$ as $\mathbb{Z}/2$-space is the sphere-bundle with the trivial involution; $S(L\otimes\xi)$ is the same space, but endowed with the antipodal action of $\mathbb{Z}/2$.

As usual, let $J(X)$ be the quotient of $KO(X)$, the real K-theory of X, by the subgroup generated by differences $[\zeta]-[\zeta']$ of vector bundles ζ, ζ' over X whose sphere-bundles are fibre-homotopy equivalent; so $J(\text{point}) = \mathbb{Z}$. $J_{\mathbb{Z}/2}(X)$ is defined similarly as the quotient of $KO_{\mathbb{Z}/2}(X)$ by differences of $\mathbb{Z}/2$-vector bundles whose sphere-bundles are equivariantly fibre-homotopy equivalent. Two $\mathbb{Z}/2$-vector bundles ζ and ζ' define the same class in $J_{\mathbb{Z}/2}(X)$ if and only if their sphere-bundles are stably $\mathbb{Z}/2$-fibre-homotopy equivalent, that is, if $S(\zeta\oplus W)$ and $S(\zeta'\oplus W)$ are equivariantly fibre-homotopy equivalent for some real representation W of $\mathbb{Z}/2$. (The same symbol W will often be used for a vector space and the corresponding trivial bundle $X\times W$ over X.)

The proposition (1.1) states that the linear map $KO(X) \longrightarrow KO_{\mathbb{Z}/2}(X)$ taking $[\xi]$ to $[L.\xi]$ - we shall often write $L.\xi$ for the tensor product $L\otimes\xi$ - induces a map of quotients $J(X) \longrightarrow J_{\mathbb{Z}/2}(X)$. The proof is simply the observation that $\xi\oplus L.\xi$ may be identified with the sum $\xi\oplus\xi$ equipped with the involution which interchanges the factors (by mapping (u,v) to (u+v,u-v) in each fibre).

More formally, let us define "doubling operations"

$$S^2 : KO(X) \longrightarrow KO_{\mathbb{Z}/2}(X) \quad\text{and}\quad S^2 : J(X) \longrightarrow J_{\mathbb{Z}/2}(X)$$

to be the linear maps taking the class of a vector bundle ξ (or sphere-bundle $S(\xi)$) to the sum $\xi\oplus\xi$ (or fibre-wise join $S(\xi) * S(\xi)$)

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More information

with the switching involution. Let

$$\sigma : KO(X) \longrightarrow KO_{\mathbb{Z}/2}(X) \quad \text{and} \quad \sigma : J(X) \longrightarrow J_{\mathbb{Z}/2}(X)$$

be the inclusions of direct summands given by regarding any bundle as a $\mathbb{Z}/2$-bundle with the trivial involution. Write

$$\bar{S}^2(x) := S^2(x) - \sigma(x).$$

Then the proposition is proved by the commutativity of the diagram:

$$\begin{array}{ccc} KO(X) & \xrightarrow{\ \bar{S}^2\ } & KO_{\mathbb{Z}/2}(X) \\ \downarrow & & \downarrow \\ J(X) & \xrightarrow[\ \bar{S}^2\ ]{} & J_{\mathbb{Z}/2}(X) . \end{array}$$

There is an immediate corollary.

Corollary (1.2). In addition to the hypotheses of (1.1), suppose that λ is a real line bundle over X. Then the sphere-bundles $S(\lambda \otimes \xi)$ and $S(\lambda \otimes \xi')$ are stably fibre-homotopy equivalent.

It is the explicit geometric construction of the operation S^2 upon which the proof of (1.1) depends; in succeeding paragraphs, particularly §§ 3-5, it will be a central theme. The mere definition of the operation lies at a different level. S^2 is simply induction

$$i_* : KO(X) \longrightarrow KO_{\mathbb{Z}/2}(X)$$

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corresponding to the inclusion $i : 0 \longrightarrow \mathbb{Z}/2$ of the trivial subgroup. (The definition is reviewed in (5.3).) It is the restriction to the diagonal of an external operation

$$S^2 : KO(X) \longrightarrow KO_{\mathbb{Z}/2}(X \times X)$$

– $X \times X$ with, as always, the switching involution – defined as the composition

$$KO(X) \xrightarrow[p_1{}^*]{} KO(X \times X) \xrightarrow[i_*]{} KO_{\mathbb{Z}/2}(X \times X),$$

where $p_1 : X \times X \longrightarrow X$ is the projection onto the first factor. In this form the definition extends to other generalized cohomology theories, in particular to stable cohomotopy.

As is implicit in the notation, a sum operation S^k may be defined for any natural number k. Write the permutation representation $\mathbb{R}^k$ of the symmetric group $\mathfrak{S}_k$ as the direct sum $\mathbb{R} \oplus V_k$ of a trivial summand and an irreducible representation of dimension k-1. The statement (1.1) clearly remains true if we substitute V_k for $L = V_2$ and $\mathfrak{S}_k$ for $\mathbb{Z}/2 = \mathfrak{S}_2$.

In later paragraphs, too, there will sometimes be generalizations, perhaps from $\mathbb{Z}/2$ to $\mathbb{Z}/p$, p an odd prime, perhaps from $\mathbb{Z}/2 = S^0$ to S^1 and S^3. We shall do no more than record the fact. $\mathbb{Z}/2$ is our proper subject.

(1.2) was first proved in answer to a question of Prof. I. M. James, and initiated much of the work described here. His own equivalent solution may be found in [41] Lemma (2.1).

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2. The Euler class and obstruction theory

$\mathbb{Z}/2$, as the subgroup $\{\pm 1\}$ of the group of units of $\mathbb{R}$, appears naturally in the study of r-fields, that is, r linearly independent cross-sections, of a real vector bundle, as soon as r is greater than 1. First, we must recall in an appropriate form the obstruction theory for a single cross-section. We need to fix a notation for stable cohomotopy, and, in view of the varying usage, do so with some care.

Notation. Let ξ and η be real vector bundles over a compact ENR X. ω^* will denote unreduced stable cohomotopy, considered as a generalized cohomology theory. A tilde indicates the associated reduced theory for pointed spaces. Thus, $\omega^*(X)$ is a graded ring with identity. Define

$$\omega^*(X;\xi-\eta) := \tilde{\omega}^*(T(\xi-\eta))$$

to be the reduced stable cohomotopy of the Thom space of the virtual bundle $\xi-\eta$. We think of the stable cohomotopy of X as a theory indexed by the category of virtual vector bundles and stable fibre-homotopy equivalences over X. If $Y \subseteq X$ is a closed sub-ENR, the relative group $\omega^*(X,Y;\xi-\eta)$ is defined to be $\tilde{\omega}^*(T(\xi-\eta)/T(\xi-\eta\,|\,Y))$. Corresponding notation is used for stable homotopy.

(In the literature $\tilde{\omega}_*$ is often written, reasonably enough as the limit of unstable homotopy groups, π_*^S (without a tilde). The coefficients '$\xi-\eta$' are sometimes written with a change of

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sign and with the dimension absorbed into the index '*'; the complexity of the indices required here precludes writing them, as is often done, as sub- or superscript.)

A superscript '+' will be used for one-point compactification, the adjunction of a base-point + if the space is already compact. ξ^+ is to be understood as the fibre-wise one-point compactification of ξ. It is a fibre-bundle over X; its fibre is a sphere with base-point. (For example, in the case that $\xi = 0$ is the zero vector bundle, 0^+ is the trivial bundle $X \times S^0$.) Then $\omega^0(X; \xi - \eta)$ may be interpreted as the group, written $\{\xi^+ ; \eta^+\}_X$, of stable fibre-homotopy classes of maps $\xi^+ \longrightarrow \eta^+$ over X preserving the base-point in each fibre (ex-maps in the terminology of [40]). (As is customary, $[-; -]$ and $\{-; -\}$ will denote respectively the set of homotopy and the group of stable homotopy classes of maps between pointed spaces.)

The way is prepared for the basic definition of obstruction theory.

Definition (2.1). The Euler class of the vector bundle ξ is the class

$$\gamma(\xi) \in \omega^0(X; -\xi) = \{0^+; \xi^+\}_X$$

represented by the inclusion $0^+ \longrightarrow \xi^+$ of the zero section (induced by $0 \subseteq \xi$).

Clearly $\gamma(\xi)$ vanishes if ξ admits a nowhere-zero cross-section. The converse is true in the 'metastable range'. Before describing

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this result, we list some elementary formal properties of the Euler class.

Proposition (2.2). Let ξ, ξ' be real vector bundles over X.

(i) Naturality. If $f: X' \longrightarrow X$ is a map, then $\gamma(f^*\xi) = f^*\gamma(\xi)$.

(ii) Multiplicativity. $\gamma(\xi \oplus \xi') = \gamma(\xi).\gamma(\xi')$.

(iii) Suppose that $a: \xi'^+ \longrightarrow \xi^+$ is a fibre map which is 'polar' in the sense that 0 is mapped to 0, + to + in each fibre. It defines a stable class in $\{\xi'^+; \xi^+\}_X = \omega^0(X; \xi' - \xi)$. Then $\gamma(\xi) = [a].\gamma(\xi')$.

The stable cohomotopy Euler class is defined by exact analogy with the classical definition in cohomology. Indeed, the group $\omega^0(D\xi, S\xi; -\xi)$ – the stable cohomotopy of the disc modulo the sphere with coefficients in (the pullback of) ξ – is naturally isomorphic to $\omega^0(X)$, the isomorphism given by a tautological 'Thom class' $u \in \omega^0(D\xi, S\xi; -\xi)$. $\gamma(\xi)$ is just the restriction of u to the zero-section $(X, \emptyset) \subseteq (D\xi, S\xi)$. More generally, if s is a cross-section of $S(\xi)$ over a closed sub-ENR Y, the <u>relative Euler class</u> $\gamma(\xi, s) \in \omega^0(X, Y; -\xi)$ is defined to be $\tilde{s}^*(u)$ for any extension $\tilde{s}: (X, Y) \longrightarrow (D\xi, S\xi)$ of the section s.

If t is another section of $S(\xi)$ over Y agreeing with s on a closed sub-ENR $Z \subseteq Y$, define their <u>difference class</u> $\delta(s,t) \in \omega^{-1}(Y, Z; -\xi) = \omega^0((Y,Z) \times (I, \dot{I}); -\xi)$ to be the relative Euler class of the pullback of ξ to $Y \times I$ with respect to the section over $Y \times \dot{I} \cup Z \times I$ which agrees with s on $Y \times 0$, t on $Y \times 1$ and their common value on $Z \times I$. It is an obstruction to deforming t into s (by a homotopy constant on Z) and determines the variation of the relative Euler class with the choice of

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cross-section.

Proposition (2.3). The difference class $\delta(s,t) \in \omega^{-1}(Y,Z;-\xi)$ is mapped to $\gamma(\xi,s) - \gamma(\xi,t) \in \omega^{0}(X,Y;-\xi)$ by the connecting homomorphism in the stable cohomotopy exact sequence of the triple (X,Y,Z).

The fundamental result in the subject is a straightforward corollary of Freudenthal's suspension theorem (proved for a cell complex X and subcomplex Y step by step over the cells).

Proposition (2.4). Suppose that the dimensions $\dim X \leq m$ and $\dim \xi = n$ lie in the metastable range: $m < 2(n-1)$.
(i) A section s of $S(\xi)$ over Y extends over the whole of X if and only if $\gamma(\xi,s) \in \omega^{0}(X,Y;-\xi)$ vanishes.
(ii) If s is a section of $S(\xi)$ over X, d an element of $\omega^{-1}(X,Y;-\xi)$, then there is a section t over X coinciding with s on Y and such that $\delta(s,t) = d$.

This leads at once to a classification theorem.

Proposition (2.5). Suppose that $m+1 < 2(n-1)$ and that $S(\xi)$ has a cross-section s. Then the set of fibre-homotopy classes of cross-sections t of $S(\xi)$ extending $s|Y$ over Y – the homotopies understood to be constant on Y – is in 1-1 correspondence with $\omega^{-1}(X,Y;-\xi)$ under the map $t \mapsto \delta(s,t)$. (The map is surjective if $m+1 \leq 2(n-1)$.)

This simple device of stabilization, formalized in (2.4) and (2.5), has both conceptual and practical advantages.

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We turn to the question of r-fields, beginning with the local classification problem. Let U and V($\neq$ 0) be real (Euclidean) vector spaces and write $O(V, U\oplus V)$ for the Stiefel manifold of isometric linear maps $V \longrightarrow U\oplus V$ with base-point, j say, the inclusion of the second factor.

Now an element of $O(V, U\oplus V)$ defines, by restriction, a map of spheres $S(V) \longrightarrow S(U\oplus V)$ commuting with the antipodal map, that is, a $\mathbb{Z}/2$-equivariant map $S(L.V) \longrightarrow S(L.(U\oplus V))$ or an equivariant cross-section of the trivial sphere-bundle $S(L.(U\oplus V))$ over the sphere $S(L.V)$. And so we are led to consider the relative Euler class and difference class in $\mathbb{Z}/2$-equivariant stable cohomotopy. The equivariant theory is indicated by a subscript; its definition, [77], is recalled in (4.1).

Definition (2.6). The <u>local</u> <u>obstruction</u>

$$\theta : \quad [X/Y; \quad O(V, U\oplus V)] \longrightarrow \omega^{-1}_{\mathbb{Z}/2}((X,Y) \times S(L.V); -L.(U\oplus V))$$

is defined as follows. A map of pairs $v: (X,Y) \longrightarrow (O(V,U\oplus V), j)$ gives, as above, a $\mathbb{Z}/2$-equivariant cross-section, v' say, of the trivial bundle $L.(U\oplus V)$ over $X \times S(L.V)$. Set $\theta(v) := \delta(v', j')$. (It clearly depends only on the homotopy class of v.)

Recall that if G is a compact Lie group and $P \longrightarrow B$ a principal G-bundle, B a compact ENR, then, just as $KO_G(P)$ is identified with $KO(B)$, [76], so the G-equivariant stable cohomotopy $\omega^*_G(P)$ is identified with $\omega^*(B)$. More generally, if E is a (virtual) G-module, then $\omega^*_G(P; E)$ is identified with $\omega^*(B; P \times_G E)$ - coefficients in the associated vector bundle over B.

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Since $\mathbb{Z}/2$ acts freely on $S(L.V)$, the target group of θ may be rewritten as $\omega^{-1}((X,Y) \times P(V); -H.(U\oplus V))$, where H, associated to the representation L, is the Hopf line bundle $(S(L.V) \times L)/\mathbb{Z}/2$ over the projective space $P(V)$. (Had we not wished to stress the equivariant theory, we might have proceeded directly to this step by noticing that an element of the Stiefel manifold determines a cross-section of $H.(U\oplus V)$ over $P(V)$.) This group is canonically isomorphic by S-duality to

$$\{X/Y;\ P(U\oplus V)/P(U)\}$$

On the other hand, the stunted projective space $P(U\oplus V)/P(U)$ is included in a standard way in the Stiefel manifold $O(V,U\oplus V)$ by the 'reflection map' R (which takes a line in $U\oplus V$ to the reflection in the orthogonal hyperplane).

Proposition (2.7). The composition $\theta.R$

$$[X/Y;\ P(U\oplus V)/P(U)] \longrightarrow [X/Y;\ O(V,U\oplus V)] \longrightarrow \{X/Y;\ P(U\oplus V)/P(U)\}$$

is the stabilization map.

The proof is by inspection.

(It is clearly enough to consider the case $(X,Y) = (P(U\oplus V),P(U))$ and look at the image of the map which collapses $P(U)$ to a point. The argument is best described in geometric language (as in §5) and for clarity we assume $U = 0$.

For any closed manifold X there is a duality isomorphism $\{X^+;\ X^+\} \cong \omega^0(X \times X; -\tau_2 X)$, where $\tau_2 X$ is the tangent bundle on

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