

# Index

- addition of matrices, 11  
 adjoint, 114  
 adjugate, 114  
 affine subset, 156, 159  
   plane, 163  
 algebra of matrices, 14–16  
 algebraic multiplicity, 269–272  
   and geometric multiplicity, 269  
   definition, 269  
 angle between vectors, 30  
 argument, 394  
 associated homogeneous system, 78  
 augmented matrix, 63
- basis, 181–185  
   change of, 225  
   definition, 181  
   extending, 190  
   of column space, 194  
   of null space, 193  
   of row space, 192  
   standard, 177
- Cauchy–Schwarz inequality, 315  
 change of basis, 225, 230, 261  
   as transformation, 226  
 change of coordinates, 223–229  
 change of variable  
   difference equations, 286  
   differential equations, 299  
 characteristic equation, 248  
 characteristic polynomial  
   definition, 248  
   of similar matrices, 262  
 coefficient matrix, 61  
 coefficients, 59  
 cofactors, 99  
   and inverse, 116  
   expansion, 100  
   matrix of, 114  
 collinear, 57  
 column space, 140, 161, 191  
 column vector, 23  
 complex conjugate, 390
- complex conjugate of matrix, 399  
 complex inner product, 401–404  
   definition, 403  
 complex inner product space, 403  
 complex matrices, 399–420  
 complex matrix, 399  
   column space, 399  
   eigenvalue, 399  
   eigenvector, 399  
   null space, 399  
   range, 399  
 complex number  
   imaginary part, 390  
   real part, 390  
 complex numbers, 389–397  
   addition, 390  
   algebra of, 390–391  
   argument, 394  
   DeMoivre’s theorem, 395  
   division, 391  
   Euler’s theorem, 396  
   exponential form, 396  
   modulus, 394  
   multiplication, 390  
   polar form, 394  
   principal argument, 394  
   viewed geometrically, 393  
 complex plane, 393  
 complex vector space, 398–399  
 composition of transformations, 217  
 conic sections, 232, 351–355  
 conjugate pairs, 392  
 consistent linear system, 69  
 consumption matrix, 120  
 coordinates, 185–186  
   change of, 225  
 coplanar, 127  
 Cramer’s rule, 117  
 cross product, *see* vector product
- definiteness  
   and eigenvalues, 342  
   and principal minors, 345, 346  
   test, 345, 346
- demand vector, 121  
 DeMoivre’s theorem, 395  
 determinant, 18, 98–113  
    $2 \times 2$ , 18  
   as sum of signed products, 103  
   effect of row operations, 110  
   of a product, 112  
   through cofactors, 99  
   through row operations, 106  
 diagonal of a matrix, 11  
 diagonalisable  
   characterisation, 271  
 diagonalisable matrix, 256  
 diagonalisation  
   and change of basis, 261  
   and eigenvectors, 257–260  
   and matrix powers, 280  
   definition, 256  
   geometrical view, 260–262  
   orthogonal, 329  
   unitary, 412  
 difference equations, 282  
   system, 282–290  
 differential equations, 296–303  
   linear system, 297  
 dimension, 186–191  
   definition, 188  
 dimension theorem  
   matrices, 198  
   transformations, 222  
 direct sum, 365  
   and orthogonal complement, 368  
 distribution vector, 292  
 dot product, 25
- echelon form, 66  
 eigenspace, 252  
 eigenvalues, 247–256  
   and definiteness, 342  
   and determinant, 253  
   and trace, 254  
   definition, 247  
   distinct, 265  
   finding, 247–252

- eigenvalues (*cont.*)  
 of complex matrix, 399  
 of Hermitian matrix, 409  
 of similar matrices, 263  
 of symmetric matrix, 332
- eigenvectors, 247–256  
 definition, 247  
 finding, 247–252  
 of complex matrix, 399  
 of similar matrices, 263  
 of symmetric matrix, 332
- elementary matrix  
 definition, 92
- elementary product, 103
- elementary row operations, 64
- ellipse, 232
- equivalence relation, 94
- Euler's formula, 396
- exponential form of complex number,  
 396
- full column rank, 372
- functions  
 pointwise addition, 150  
 pointwise scalar multiplication, 150
- Fundamental Theorem of Algebra, 391
- Gauss–Jordan elimination, 64
- Gaussian elimination, 64–78, 133–139  
 leading variables, 72  
 non-leading variables, 72
- geometric multiplicity, 269–272  
 and algebraic multiplicity, 269  
 definition, 269
- geometry of vectors, 27–33
- Gram–Schmidt process, 321–323
- Hermitian conjugate, 407
- Hermitian matrix, 408  
 has real eigenvalues, 409  
 orthogonal eigenvectors, 410
- homogeneous system, 75  
 associated, 78
- homogeneous systems  
 are consistent, 75
- hyperplane, 47, 164, 188
- idempotent, 374
- identity matrix, 15
- identity transformation, 216
- imaginary number, *see* complex numbers
- imaginary part  
 complex number, 390
- indefinite quadratic form, 341
- initial condition, 282, 284, 297
- inner product, 24, 312–314  
 and angle, 30, 316  
 and length, 30, 315  
 and norm, 315, 403  
 and orthogonality, 32, 404  
 and unitary matrix, 411  
 Cauchy–Schwarz inequality, 315  
 complex, 403  
 definition, 313  
 Euclidean, 312  
 geometrical interpretation, 30–32  
 properties, 25  
 standard, 24, 312
- inner product space, 313  
 complex, 403
- input–output analysis, 119
- inverse  
 of linear transformation, 218
- inverse matrix, 16–20, 94–98,  
 113–118  
 $2 \times 2$ , 18  
 by cofactors, 116  
 definition, 17  
 is unique, 17
- inverse of product, 20
- invertible matrix, 18
- isometry, 355
- kernel  
 of linear transformation, 220  
 of matrix, 78
- leading variables, 72
- least squares solution, 380–383
- length of a vector, 29
- Leontief input–output analysis, 119
- linear combination, 24, 140, 153, 160  
 closure under, 158  
 non-trivial, 172  
 uniqueness, 175
- linear dependence, 172–181
- linear equations, *see* linear system
- linear function, *see* linear transformation
- linear independence, 172–181  
 test, 175–179
- linear mapping, 210
- linear operator, 210
- linear span, 160
- linear system, 59–78  
 augmented matrix for, 63  
 coefficient matrix, 61  
 coefficients, 59  
 consistent, 69  
 general solution, 73  
 geometrical interpretation, 69  
 homogeneous, 75  
 infinitely many solutions, 73  
 of difference equations, 282–290  
 of differential equations, 297  
 particular solution, 73
- Principle of Linearity, 79
- rank, 135–139
- row operations, 64
- solution of, 60
- solution set, 74
- vector form of solution, 73
- vector solution, 138
- linear transformation, 210–223  
 and matrices, 212–216, 220  
 change of basis, 230  
 composition, 217  
 corresponding to matrix, 212  
 definition, 210  
 dimension theorem, 222  
 examples, 211–212  
 idempotent, 374  
 identity, 216  
 inverse, 218  
 kernel, 220  
 matrix of, 213, 220, 230  
 null space, 220  
 range, 220  
 rank, 221  
 rank–nullity theorem, 222  
 rotation, 215  
 zero, 216
- lines, 33–39, 162, 188  
 Cartesian equation, 33, 37  
 coincident, 37  
 coplanar, 38  
 skew, 38  
 vector equation, 33–36
- long-term distribution, 293
- Markov chain, 290–296  
 distribution vector, 292  
 long-term distribution, 293  
 regular, 294  
 state vector, 292  
 transition matrix, 292
- Markov process, 291
- matrix, 10  
 addition, 11  
 additive identity, 15  
 associative, 14  
 commutative, 14  
 algebra, 14–16  
 associativity, 14  
 distributivity, 15  
 characteristic polynomial, 248  
 cofactors, 99  
 column space, 161, 191  
 complex conjugate, 399  
 corresponding transformation, 212  
 definition of, 10  
 determinant, *see* determinant  
 diagonalisable, 256  
 dimension theorem, 198

- eigenvalues of, 247
- eigenvectors of, 247
- elementary, 92
- entry, 10
- equality, 11
- Hermitian, 408
- Hermitian conjugate, 407
- idempotent, 374
- identity, 15
- ill-conditioned, 89
- inverse, 16–20, 94–98, 113–118
  - $2 \times 2$ , 18
  - definition, 17
  - is unique, 17
  - of product, 20
- invertible, 18
- kernel of, 78
- minor, 99
- multiplication, 12–14
  - associative, 14
  - multiplicative identity, 15
  - non-commutative, 13
- multiplication by a scalar, 11
- non-invertible, 18
- non-singular, 18
- normal, 412
- null space, 78, 158, 191, 220
- nullity, 195
- of cofactors, 114
- of linear transformation, 213, 220
- of quadratic form, 340
- of rotation, 215
- orthogonal, 319
- range, 139, 220
- rank, 132, 195
- rank–nullity theorem, 198
- row equivalence, 94
- row space, 161, 191
- singular, 18
- size, 10
- skew symmetric, 56
- skew-Hermitian, 429
- square, 11
- stochastic, 295
- symmetric, 22
- trace, 254
- transpose, 20
  - definition, 21
  - properties, 21–22
- unitary, 410
- zero, 15
- matrix diagonal, 11
- matrix inverse
  - of product, 20
- matrix power, 279
- matrix powers, 20
  - and spectral decomposition, 418
  - via diagonalisation, 280
- minor, 99
  - principal, 344
- modulus, 394
- multiplication of matrices, 12–14
- multiplicity
  - algebraic, 269
  - geometric, 269
- negative definite, 341
- negative semi-definite, 341
- non-diagonalisable matrices, 264
- non-invertible matrix, 18
- non-leading variables, 72
- non-singular matrix, 18
- non-trivial linear combination, 172
- norm
  - definition, 315, 403
- normal matrix, 412
- normal vector, 41
- normalisation, 315
- null space, 191
  - matrix, 158
  - of linear transformation, 220
  - of matrix, 78, 220
  - orthogonal complement, 370
- nullity
  - of matrix, 195
- orthogonal
  - vectors, 32, 317, 404
- orthogonal complement, 367–372
  - and direct sum, 368
  - definition, 367
  - is a subspace, 367
  - of null space, 370
  - or range, 370
- orthogonal diagonalisation, 329
  - of symmetric matrices, 329–339
  - Spectral theorem, 331
- orthogonal matrix
  - definition, 319
  - has orthonormal columns, 321
- orthogonal projection, 374
  - idempotent and symmetric, 376
  - onto range, 377
- orthogonality, 316–320
  - and inner product, 32, 404
  - and linear independence, 318
- orthonormal basis, 320
- orthonormal set
  - and orthogonal matrix, 321
  - and unitary matrix, 410
  - definition, 320, 404
- orthonormalisation, 321–323
- permutation, 102
  - inversion, 102
- planes, 39–46, 162, 188
- affine subsets, 163
- as subspaces, 162
- Cartesian equation, 41–46, 184
- normal vector to, 41
- parametric equation, 39
- vector equation, 39
- polar form of complex number, 394
- portfolio, 88
  - arbitrage, 89
  - riskless, 88
- positive definite, 341
- positive semi-definite, 341
- power of a matrix, 20
- powers of a matrix, 279–281
  - and difference equations, 284–286
  - definition, 279
- principal argument, 394
- principal minor, 344
  - and definiteness, 345, 346
- Principle of Linearity, 79
- production vector, 119
- projection, 372–378
  - definition, 372
  - idempotent, 375
  - orthogonal, 374
- Pythagoras’ theorem
  - generalised, 317
- quadratic form
  - definition, 340
  - indefinite, 341
  - negative definite, 341
  - negative semi-definite, 341
  - positive definite, 341
  - positive semi-definite, 341
- quadratic forms, 339–355
- range
  - and column space, 162
  - of linear transformation, 220
  - of matrix, 139, 220
  - orthogonal complement, 370
- rank
  - and linear system, 135–139
  - full column, 372
  - of matrix, 132, 195
  - of transformation, 221
- rank–nullity theorem
  - matrices, 198
  - transformations, 222
- real part
  - complex number, 390
- RREF, 67
- reduced row echelon form, 67
- regular Markov chain, 294
- rigid motion, 355
- rotation, 215, 353
- row echelon form, 66

- row equivalence, 94
- row operations, 62–64
  - and determinants, 110
- row space, 161, 191
- row vector, 23
  
- scalar, 11
- scalar product, 25
- scalar triple product, 127
- signed elementary product, 103
- similarity, 231–235, 256
  - and characteristic polynomial, 262
  - and eigenvalues, 263
  - and eigenvectors, 263
- simultaneous equations, 60
- singular matrix, 18
- size of a matrix, 10
- skew symmetric matrix, 56
- solution set of linear system, 74
- spectral decomposition, 415–420
  - and matrix powers, 418
- Spectral theorem, 331
  - proof, 337–339
- square matrix, 11
- standard basis, 177, 182
- state price, 89
- state vector, 292
- stochastic matrix, 295
- subspace
  - definition, 154
  - direct sum, 365
  
- examples, 155–157
- planes, 162
- spanned by a set, 160
- sum, 364
- test for, 157
- sum of subspaces, 364
- symmetric matrix, 22
  - and quadratic form, 340
  - eigenvalues, 332
  - eigenvectors, 332
  - orthogonal diagonalisation, 329–339
- system
  - of equations, *see* linear system
  
- technology matrix, 120
- test of definiteness, 345, 346
- trace, 254
- transition matrix
  - of basis, 225
  - or Markov chain, 292
- transpose, 20
  - definition, 21
  - properties, 21–22
- triangle inequality, 317
- trivial solution, 75
  
- unit vector, 30
- unitary diagonalisation, 412
- unitary matrix, 410
  - and inner product, 411
  - has orthonormal columns, 410
  
- vector, 23
  - addition, 23
  - column, 23
  - components of, 23
  - definition, 23
  - direction of, 30
  - dot product, 25
  - entries of, 23
  - geometrical interpretation, 27–33
  - inner product, 24
  - length of, 29
  - linear combination, 24
  - orthogonal, 32, 317, 404
  - parallel, 30
  - perpendicular, 32
  - row, 23
  - scalar multiplication, 23
  - scalar product, 25
  - zero, 24
- vector product, 127
- vector space
  - axioms, 150
  - definition, 150
  - examples of, 151
  - finite-dimensional, 188
  - infinite-dimensional, 188
  - subspace, 154
  
- zero matrix, 15
- zero transformation, 216
- zero vector, 24