

Linear Algebra: Concepts and Methods

Any student of linear algebra will welcome this textbook, which provides a thorough treatment of this key topic. Blending practice and theory, the book enables the reader to learn and comprehend the standard methods, with an emphasis on understanding how they actually work. At every stage, the authors are careful to ensure that the discussion is no more complicated or abstract than it needs to be, and focuses on the most fundamental topics.

- Hundreds of examples and exercises, including solutions, give students plenty of hands-on practice
- End-of-chapter sections summarise material to help students consolidate their learning
- Ideal as a course text and for self-study
- Instructors can use the many examples and exercises to supplement their own assignments
- Both authors have extensive experience of undergraduate teaching and of preparation of distance learning materials.

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To Colleen, Alistair, and my parents. And, just for Alistair,
here's one of those sideways moustaches: }
(MA)

To Bill, for his support throughout, and to my father, for his
encouragement to study mathematics.
(MH)

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Preface

Linear algebra is one of the core topics studied at university level by students on many different types of degree programme. Alongside calculus, it provides the framework for mathematical modelling in many diverse areas. This text sets out to introduce and *explain* linear algebra to students from any discipline. It covers all the material that would be expected to be in most first-year university courses in the subject, together with some more advanced material that would normally be taught later.

The book has drawn on our extensive experience over a number of years in teaching first- and second-year linear algebra to LSE undergraduates and in providing self-study material for students studying at a distance. This text represents our best effort at distilling from our experience what it is that we think works best in helping students not only to *do* linear algebra, but to *understand* it. We regard *understanding* as essential. ‘Understanding’ is not some fanciful intangible, to be dismissed because it does not constitute a ‘demonstrable learning outcome’: it is at the heart of what higher education (rather than merely more education) is about. Linear algebra is a coherent, and beautiful, part of mathematics: manipulation of matrices and vectors leads, with a dash of abstraction, to the underlying concepts of vector spaces and linear transformations, in which contexts the more mechanical, manipulative, aspects of the subject *make sense*. It is worth striving for understanding, not only because of the inherent intellectual satisfaction, but because it pays off in other ways: it helps a student to work with the methods and techniques because he or she knows *why* these work and *what they mean*.

Large parts of the material in this book have been adapted and developed from lecture notes prepared by MH for the Mathematical Methods course at the LSE, a long-established course which has a large audience, and which has evolved over many years. Other parts have been influenced by MA’s teaching of non-specialist first-year courses and second-year linear algebra. Both of us have written self-study materials for

students; some of the book is based on material originally produced by us for the programmes in economics, management, finance and the social sciences by distance and flexible learning offered by the University of London International Programmes (www.londoninternational.ac.uk).

We have attempted to write a user-friendly, fairly interactive and helpful text, and we intend that it could be useful not only as a course text, but for self-study. To this end, we have written in what we hope is an open and accessible – sometimes even conversational – style, and have included ‘learning outcomes’ and many ‘activities’ and ‘exercises’. We have also provided a very short introduction just to indicate some of the background which a reader should, ideally, possess (though if some of that is lacking, it can easily be acquired in passing).

Reading a mathematics book properly cannot be a passive activity: the reader should interrogate the text and have pen and paper at the ready to check things. To help in this, the chapters contain many activities – prompts to a reader to be an ‘active’ reader, to pause for thought and really make sure they understand what has just been written, or to think ahead and anticipate what is to come next. At the end of chapters, there are comments on most of the activities, which a reader can consult to confirm his or her understanding.

The main text of each chapter ends with a brief list of ‘learning outcomes’. These are intended to highlight the main aspects of the chapter, to help a reader review and consolidate what has been read.

There are carefully designed exercises towards the end of each chapter, with full solutions (not just brief answers) provided at the end of the book. These exercises vary in difficulty from the routine to the more challenging, and they are one of the key ingredients in helping a reader check his or her understanding of the material. Of course, these are best made use of by attempting them seriously before consulting the solution. (It’s all very easy to read and agree with a solution, but unless you have truly grappled with the exercise, the benefits of doing so will be limited.)

We also provide sets of additional exercises at the end of each chapter, which we call Problems as the solutions are not given. We hope they will be useful for assignments by teachers using this book, who will be able to obtain solutions from the book’s webpage. Students will gain confidence by tackling, and solving, these problems, and will be able to check many of their answers using the techniques given in the chapter.

Over the years, many people – students and colleagues – have influenced and informed the way we approach the teaching of linear algebra, and we thank them all.