Preliminaries: before we begin

This short introductory chapter discusses some very basic aspects of mathematics and mathematical notation that it would be useful to be comfortable with before proceeding. We imagine that you have studied most (if not all) of these topics in previous mathematics courses and that nearly all of the material is revision, but don't worry if a topic is new to you. We will mention the main results which you will need to know. If you are unfamiliar with a topic, or if you find any of the topics difficult, then you should look up that topic in any basic mathematics text.

Sets and set notation

A set may be thought of as a collection of objects. A set is usually described by listing or describing its *members* inside curly brackets. For example, when we write $A = \{1, 2, 3\}$, we mean that the objects belonging to the set A are the numbers 1, 2, 3 (or, equivalently, the set A consists of the numbers 1, 2 and 3). Equally (and this is what we mean by 'describing' its members), this set could have been written as

$$A = \{n \mid n \text{ is a whole number and } 1 \leq n \leq 3\}.$$

Here, the symbol | stands for 'such that'. (Sometimes, the symbol ':' is used instead.) As another example, the set

$$B = \{x \mid x \text{ is a reader of this book}\}$$

has as its members all of you (and nothing else). When x is an object in a set A, we write $x \in A$ and say 'x belongs to A' or 'x is a member of A'.

The set which has no members is called the *empty set* and is denoted by $\emptyset$. The empty set may seem like a strange concept, but it has its uses.

We say that the set S is a *subset* of the set T, and we write $S \subseteq T$, or $S \subset T$, if every member of S is a member of T. For example, $\{1, 2, 5\} \subseteq \{1, 2, 4, 5, 6, 40\}$. The difference between the two symbols is that $S \subset T$ means that S is a *proper subset* of T, meaning not all of T, and $S \subseteq T$ means that S is a subset of T and possibly (but not necessarily) all of T. So in the example just given we could have also written $\{1, 2, 5\} \subset \{1, 2, 4, 5, 6, 40\}$.

Given two sets A and B, the *union* $A \cup B$ is the set whose members belong to A or B (or both A and B); that is,

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

For example, if $A = \{1, 2, 3, 5\}$ and $B = \{2, 4, 5, 7\}$, then $A \cup B = \{1, 2, 3, 4, 5, 7\}$.

Similarly, we define the *intersection* $A \cap B$ to be the set whose members belong to both A and B:

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

So, if $A = \{1, 2, 3, 5\}$ and $B = \{2, 4, 5, 7\}$, then $A \cap B = \{2, 5\}$.

Numbers

There are some standard notations for important sets of numbers. The set $\mathbb{R}$ of *real numbers*, the 'normal' numbers you are familiar with, may be thought of as the points on a line. Each such number can be described by a decimal representation.

The set of real numbers $\mathbb{R}$ includes the following subsets: $\mathbb{N}$, the set of natural numbers, $\mathbb{N} = \{1, 2, 3, \dots\}$, also referred to as the positive integers; $\mathbb{Z}$, the set of all integers, $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$; and $\mathbb{Q}$, the set of rational numbers, which are numbers that can be written as fractions, p/q, with $p, q \in \mathbb{Z}$, $q \neq 0$. In addition to the real numbers, there is the set $\mathbb{C}$ of *complex* numbers. You may have seen these before, but don't worry if you have not; we cover the basics at the start of Chapter 13, when we need them.

The *absolute value* of a real number a is defined by

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a \leq 0 \end{cases}.$$

So the absolute value of a equals a if a is non-negative (that is, if $a \geq 0$), and equals $-a$ otherwise. For instance, $|6| = 6$ and $|-2.5| = 2.5$. Note that

$$\sqrt{a^2} = |a|,$$

since by $\sqrt{x}$ we always mean the non-negative square root to avoid ambiguity. So the two solutions of the equation $x^2 = 4$ are $x = \pm 2$ (meaning $x = 2$ or $x = -2$), but $\sqrt{4} = 2$.

The absolute value of real numbers satisfies the following inequality:

$$|a + b| \leq |a| + |b|, \qquad a, b \in \mathbb{R}.$$

Having defined $\mathbb{R}$, we can define the set $\mathbb{R}^2$ of *ordered pairs* (x, y) of real numbers. Thus, $\mathbb{R}^2$ is the set usually depicted as the set of points in a plane, x and y being the coordinates of a point with respect to a pair of axes. For instance, $(-1, 3/2)$ is an element of $\mathbb{R}^2$ lying to the left of and above $(0, 0)$, which is known as the *origin*.

Mathematical terminology

In this book, as in most mathematics texts, we use the words ‘definition’, ‘theorem’ and ‘proof’, and it is important not to be daunted by this language if it is unusual to you. A definition is simply a precise statement of what a particular idea or concept means. Definitions are hugely important in mathematics, because it is a precise subject. A theorem is just a statement or result. A proof is an explanation as to why a theorem is true. As a fairly trivial example, consider the following:

Definition: An integer n is *even* if it is a multiple of 2; that is, if $n = 2k$ for some integer k.

Note that this is a precise statement *telling us* what the word ‘even’ means. It is *not* to be taken as a ‘result’: it’s defining what the word ‘even’ means.

Theorem: *The sum of two even integers is even. That is, if m, n are even, so is $m + n$.*

Proof: Suppose m, n are even. Then, by the definition, there are integers k, l such that $m = 2k$ and $n = 2l$. Then

$$m + n = 2k + 2l = 2(k + l).$$

Since $k + l$ is an integer, it follows that $m + n$ is even. □

Note that, as here, we often use the symbol □ to denote the end of a proof. This is just to make it clear where the proof ends and the following text begins.

Occasionally, we use the term 'corollary'. A corollary is simply a result that is a consequence of a theorem and perhaps isn't 'big' enough to be called a theorem in its own right.

Don't worry about this terminology if you haven't met it before. It will become familiar as you work through the book.

Basic algebra

Algebraic manipulation

You should be capable of manipulating simple algebraic expressions and equations.

You should be proficient in:

- collecting up terms; for example, $2a + 3b - a + 5b = a + 8b$
- multiplication of variables; for example,

$$a(-b) - 3ab + (-2a)(-4b) = -ab - 3ab + 8ab = 4ab$$

- expansion of bracketed terms; for example,

$$\begin{aligned} -(a - 2b) &= -a + 2b, \\ (2x - 3y)(x + 4y) &= 2x^2 - 3xy + 8xy - 12y^2 \\ &= 2x^2 + 5xy - 12y^2. \end{aligned}$$

Powers

When n is a positive integer, the nth *power* of the number a, denoted a^n, is simply the product of n copies of a; that is,

$$a^n = \underbrace{a \times a \times a \times \cdots \times a}_{n \text{ times}}.$$

The number n is called the *power*, *exponent* or *index*. We have the *power rules* (or *rules of exponents*),

$$a^r a^s = a^{r+s}, \quad (a^r)^s = a^{rs},$$

whenever r and s are positive integers.

The power a^0 is defined to be 1.

The definition is extended to negative integers as follows. When n is a positive integer, a^{-n} means $1/a^n$. For example, 3^{-2} is $1/3^2 = 1/9$. The power rules hold when r and s are any integers, positive, negative or zero.

When n is a positive integer, $a^{1/n}$ is the positive nth root of a; this is the positive number x such that $x^n = a$. For example, $a^{1/2}$ is usually denoted by $\sqrt{a}$, and is the positive *square root* of a, so that $4^{1/2} = 2$.

When m and n are integers and n is positive, $a^{m/n}$ is $(a^{1/n})^m$. This extends the definition of powers to the rational numbers (numbers which can be written as fractions). The definition is extended to real numbers by 'filling in the gaps' between the rational numbers, and it can be shown that the rules of exponents still apply.

Quadratic equations

It is straightforward to find the *solution* of a linear equation, one of the form $ax + b = 0$ where $a, b \in \mathbb{R}$. By a solution, we mean a real number x for which the equation is true.

A common problem is to find the set of solutions of a *quadratic* equation

$$ax^2 + bx + c = 0,$$

where we may as well assume that $a \neq 0$, because if $a = 0$ the equation reduces to a linear one. In some cases, the quadratic expression can be factorised, which means that it can be written as the product of two linear terms. For example,

$$x^2 - 6x + 5 = (x - 1)(x - 5),$$

so the equation $x^2 - 6x + 5 = 0$ becomes $(x - 1)(x - 5) = 0$. Now, the only way that two numbers can multiply to give 0 is if at least one of the numbers is 0, so we can conclude that $x - 1 = 0$ or $x - 5 = 0$; that is, the equation has two solutions, 1 and 5.

Although factorisation may be difficult, there is a general method for determining the solutions to a quadratic equation using the *quadratic formula*, as follows. Suppose we have the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$. Then the solutions of this equation are

$$x_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \qquad x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$

The term $b^2 - 4ac$ is called the *discriminant*.

- If $b^2 - 4ac > 0$, the equation has *two* real solutions as given above.
- If $b^2 - 4ac = 0$, the equation has *exactly one* solution, $x = -b/(2a)$. (In this case, we say that this is a solution of multiplicity two.)

- If $b^2 - 4ac < 0$, the equation has *no* real solutions. (It will have complex solutions, but we explain this in Chapter 13.)

For example, consider the equation $2x^2 - 7x + 3 = 0$. Using the quadratic formula, we have

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{7 \pm \sqrt{49 - 4(2)(3)}}{2(2)} = \frac{7 \pm 5}{4}.$$

So the solutions are $x = 3$ and $x = \frac{1}{2}$.

The equation $x^2 + 6x + 9 = 0$ has one solution of multiplicity 2; its discriminant is $b^2 - 4ac = 36 - 9(4) = 0$. This equation is most easily solved by recognising that $x^2 + 6x + 9 = (x + 3)^2$, so the solution is $x = -3$.

On the other hand, consider the quadratic equation

$$x^2 - 2x + 3 = 0;$$

here we have $a = 1, b = -2, c = 3$. The quantity $b^2 - 4ac$ is negative, so this equation has no real solutions. This is less mysterious than it may seem. We can write the equation as $(x - 1)^2 + 2 = 0$. Rewriting the left-hand side of the equation in this form is known as *completing the square*. Now, the square of a number is always greater than or equal to 0, so the quantity on the left of this equation is always at least 2 and is therefore never equal to 0. The quadratic formula for the solutions to a quadratic equation is obtained using the technique of completing the square. Quadratic polynomials which cannot be written as a product of linear terms (so ones for which the discriminant is negative) are said to be *irreducible*.

Polynomial equations

A polynomial of degree n in x is an expression of the form

$$P_n(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n,$$

where the a_i are real constants, $a_n \neq 0$, and x is a real variable. For example, a quadratic expression such as those discussed above is a polynomial of degree 2.

A polynomial equation of degree n has at most n solutions. For example, since

$$x^3 - 7x + 6 = (x - 1)(x - 2)(x + 3),$$

the equation $x^3 - 7x + 6 = 0$ has three solutions; namely, $1, 2, -3$. The solutions of the equation $P_n(x) = 0$ are called the *roots* or *zeros* of the polynomial. Unfortunately, there is no general straightforward formula (as there is for quadratics) for the solutions to $P_n(x) = 0$ for polynomials P_n of degree larger than 2.

To find the solutions to $P(x) = 0$, where P is a polynomial of degree n, we use the fact that if α is such that $P(\alpha) = 0$, then $(x - \alpha)$ must be a factor of $P(x)$. We find such an a by trial and error and then write $P(x)$ in the form $(x - \alpha)Q(x)$, where $Q(x)$ is a polynomial of degree $n - 1$.

As an example, we'll use this method to factorise the cubic polynomial $x^3 - 7x + 6$. Note that if this polynomial can be expressed as a product of linear factors, then it will be of the form

$$x^3 - 7x + 6 = (x - r_1)(x - r_2)(x - r_3),$$

where its constant term is the product of the roots: $6 = -r_1r_2r_3$. (To see this, just substitute $x = 0$ into both sides of the above equation.) So if there is an integer root, it will be a factor of 6. We will try $x = 1$. Substituting this value for x, we do indeed get $1 - 7 + 6 = 0$, so $(x - 1)$ is a factor. Then we can deduce that

$$x^3 - 7x + 6 = (x - 1)(x^2 + \lambda x - 6)$$

for some number λ, as the coefficient of x^2 must be 1 for the product to give x^3, and the constant term must be -6 so that $(-1)(-6) = 6$, the constant term in the cubic. It only remains to find λ. This is accomplished by comparing the coefficients of either x^2 or x in the cubic polynomial and the product. The coefficient of x^2 in the cubic is 0, and in the product the coefficient of x^2 is obtained from the terms $(-1)(x^2) + (x)(\lambda x)$, so that we must have $\lambda - 1 = 0$ or $\lambda = 1$. Then

$$x^3 - 7x + 6 = (x - 1)(x^2 + x - 6),$$

and the quadratic term is easily factorised into $(x - 2)(x + 3)$; that is,

$$x^3 - 7x + 6 = (x - 1)(x - 2)(x + 3).$$

Trigonometry

The trigonometrical functions, $\sin\theta$ and $\cos\theta$ (the *sine function* and *cosine function*), are very important in mathematics. You should know their geometrical meaning. (In a right-angled triangle, $\sin\theta$ is the ratio of the length of the side opposite the angle θ to the length of the hypotenuse, the longest side of the triangle; and $\cos\theta$ is the ratio of the length of the side adjacent to the angle to the length of the hypotenuse.)

It is important to realise that throughout this book angles are measured in *radians* rather than *degrees*. The conversion is as follows: 180 degrees equals π radians, where π is the number 3.141 . . .. It is good practice *not* to expand π or multiples of π as decimals, but to leave them in terms of the symbol π. For example, since 60 degrees is one-third of 180 degrees, it follows that in radians 60 degrees is $\pi/3$.

The sine and cosine functions are related by the fact that $\cos x = \sin(x + \frac{\pi}{2})$, and they always take a value between 1 and -1. Table 1 gives some important values of the trigonometrical functions.

There are some useful results about the trigonometrical functions, which we use now and again. In particular, for any angles θ and ϕ, we have

$$\sin^2\theta + \cos^2\theta = 1,$$
$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

and

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi.$$

Table 1

$\boldsymbol{\theta}$	**sin** $\boldsymbol{\theta}$	**cos** $\boldsymbol{\theta}$
0	0	1
$\pi/6$	$1/2$	$\sqrt{3}/2$
$\pi/4$	$1/\sqrt{2}$	$1/\sqrt{2}$
$\pi/3$	$\sqrt{3}/2$	$1/2$
$\pi/2$	1	0

A little bit of logic

It is very important to understand the formal meaning of the word ‘if’ in mathematics. The word is often used rather sloppily in everyday life, but has a very precise mathematical meaning. Let’s give an example. Suppose someone tells you ‘If it rains, then I wear a raincoat’, and suppose that this is a true statement. Well, then suppose it rains. You can certainly conclude the person will wear a raincoat. But what if it does not rain? Well, you can’t conclude anything. The statement only tells you about what happens *if* it rains. If it does not, then the person might, or might not, wear a raincoat. You have to be clear about this: an ‘if–then’ statement only tells you about what follows *if* something particular happens.

More formally, suppose P and Q are mathematical statements (each of which can therefore be either true or false). Then we can form the statement denoted $P \Longrightarrow Q$ (‘P implies Q’ or, equivalently, ‘if P, then Q’), which means ‘if P is true, then Q is true’. For instance, consider the theorem we used as an example earlier. This says that if m, n are even integers, then so is $m + n$. We can write this as

$$m, n \text{ even integers} \Longrightarrow m + n \text{ is even.}$$

The *converse* of a statement $P \Longrightarrow Q$ is $Q \Longrightarrow P$ and whether that is true or not is a separate matter. For instance, the converse of the statement just made is

$$m + n \text{ is even} \Longrightarrow m, n \text{ even integers.}$$

This is *false*. For instance, $1 + 3$ is even, but 1 and 3 are not.

If, however, both statements $P \Longrightarrow Q$ and $Q \Longrightarrow P$ are true, then we say that Q is true *if and only if* P is. Alternatively, we say that P and Q are *equivalent*. We use the single piece of notation $P \Longleftrightarrow Q$ instead of the two separate $P \Longrightarrow Q$ and $Q \Longrightarrow P$.

1

Matrices and vectors

Matrices and vectors will be the central objects in our study of linear algebra. In this chapter, we introduce matrices, study their properties and learn how to manipulate them. This will lead us to a study of vectors, which can be thought of as a certain type of matrix, but which can more usefully be viewed geometrically and applied with great effect to the study of lines and planes.

1.1 What is a matrix?

Definition 1.1 (Matrix) A matrix is a rectangular array of numbers or symbols. It can be written as

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

We denote this array by the single letter A or by (a_{ij}), and we say that A has m rows and n columns, or that it is an $m \times n$ matrix. We also say that A is a matrix of *size* $m \times n$.

The number a_{ij} in the ith row and jth column is called the (i, j) entry. Note that the first subscript on a_{ij} always refers to the row and the second subscript to the column.

Example 1.2 The matrix

$$A = \begin{pmatrix} 2 & 1 & 7 & 8 \\ 0 & -2 & 5 & -1 \\ 4 & 9 & 3 & 0 \end{pmatrix}$$