FINITE GROUPS OF LIE TYPE AND THEIR REPRESENTATIONS

GERHARD HISS

Lehrstuhl D für Mathematik, RWTH Aachen University, 52056 Aachen, Germany
Email: gerhard.hiss@math.rwth-aachen.de

This article is a slightly expanded account of the series of four lectures I gave at the conference. It is intended as a (non-comprehensive) survey covering some important aspects of the representation theory of finite groups of Lie type, where the emphasis is put on the problem of labelling the irreducible representations and of finding their degrees. All three cases are covered, representations in characteristic zero, in defining as well as in non-defining characteristics.

The first section introduces various ways of defining groups of Lie type and some classes of important subgroups of them. The next three sections are devoted to the representation theory of these groups, each section covering one of the three cases.

The lectures were addressed at a broad audience. Thus on the one hand, I have tried to introduce even the most fundamental notions, but on the other hand, I have also tried to get right to the edge of today's knowledge in the topics discussed. As a consequence, the lectures were of a somewhat inhomogeneous level of difficulty. In this article I have omitted the most introductory material. The reader may find all background material needed from representation theory in the textbook [51] by Isaacs.

For this survey I have included a few more examples, as well as most of the references to the results presented in my talks. The sections in this article correspond to the four lectures I have given, the subsections to the sections inside the lectures, and the subsubsections to the individual slides.

1 The finite groups of Lie type

In this first section we give various examples and constructions for finite groups of Lie type, we introduce the concepts of finite reductive groups and groups with BN-pairs. All of this material can be found in the books by Carter [9, 10] and Steinberg [77, 78].

1.1 Various constructions for finite groups of Lie type

One of the motivations to study finite groups of Lie type stems from the fact that this class of groups constitutes a large portion of the class of all finite simple groups.

1.1.1 The classification of the finite simple groups

"Most" finite simple groups are closely related to finite groups of Lie type. This is a consequence of the classification theorem of the finite simple groups.

Theorem 1.1 (Classification of the finite simple groups) *Every finite simple group is*

(1) *one of* 26 *sporadic simple groups; or*
(2) *a cyclic group of prime order; or*
(3) *an alternating group* A_n *with* $n \geq 5$*; or*
(4) *closely related to a finite group of Lie type.*

So what are finite groups of Lie type? A first answer could be: Finite analogues of Lie groups.

1.1.2 The finite classical groups

Examples for finite analogues of Lie groups are the finite classical groups, i.e. full linear groups or linear groups preserving a form of degree 2, defined over finite fields. Let us list a few examples of classical groups.

Example 1.2 $\mathrm{GL}_n(q)$, $\mathrm{GU}_n(q)$, $\mathrm{Sp}_{2m}(q)$, $\mathrm{SO}_{2m+1}(q)$ $\ldots$ (q a prime power) are classical groups. To be more specific, we may define

$$\mathrm{SO}_{2m+1}(q) = \{g \in SL_{2m+1}(q) \mid g^{tr} J g = J\},$$

with

$$J = \begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix} \in \mathbb{F}_q^{2m+1 \times 2m+1}.$$

Related groups, e.g. $\mathrm{SL}_n(q)$, $\mathrm{PSL}_n(q)$, $\mathrm{CSp}_{2m}(q)$, the conformal symplectic group, etc. are also classical groups.

Not all classical groups are simple, but they are closely related to simple groups. For example, the projective special linear group $\mathrm{PSL}_n(q) = \mathrm{SL}_n(q)/Z(SL_n(q))$ is simple (unless $(n, q) = (2, 2), (2, 3)$), but $\mathrm{SL}_n(q)$ is not simple in general.

1.1.3 Exceptional groups

There are groups of Lie type which are not classical, namely, the *exceptional groups* $G_2(q)$, $F_4(q)$, $E_6(q)$, $E_7(q)$, $E_8(q)$ (q a prime power), the *twisted groups* ${}^2E_6(q)$, ${}^3D_4(q)$ (q a prime power), the *Suzuki groups* ${}^2B_2(2^{2m+1})$ ($m \geq 0$), and the *Ree groups* ${}^2G_2(3^{2m+1})$ and ${}^2F_4(2^{2m+1})$ ($m \geq 0$). The names of these groups, e.g. $G_2(q)$ or $E_8(q)$ refer to simple complex Lie algebras or rather their root systems.

Some of the questions we are going to discuss in this section are: How are groups of Lie type constructed? What are their properties, subgroups, orders, etc?

1.1.4 The orders of some finite groups of Lie type

The orders of groups of Lie type are given by nice formulae.

Example 1.3 Here are these order formulae for some finite groups of Lie type.

$|\mathrm{GL}_n(q)| = q^{n(n-1)/2}(q-1)(q^2-1)(q^3-1)\cdots(q^n-1)$.
$|\mathrm{GU}_n(q)| = q^{n(n-1)/2}(q+1)(q^2-1)(q^3+1)\cdots(q^n-(-1)^n)$.
$|\mathrm{SO}_{2m+1}(q)| = q^{m^2}(q^2-1)(q^4-1)\cdots(q^{2m}-1)$.
$|F_4(q)| = q^{24}(q^2-1)(q^6-1)(q^8-1)(q^{12}-1)$.
$|{}^2F_4(q)| = q^{12}(q-1)(q^3+1)(q^4-1)(q^6+1)\ (q = 2^{2m+1})$.

Is there a systematic way to derive these formulae?

1.1.5 Root systems

We take a little detour to discuss root systems and related structures. Let V be a finite-dimensional real vector space endowed with an inner product $(-,-)$.

Definition 1.4 A root system in V is a finite subset $\Phi \subset V$ satisfying:
(1) Φ spans V as a vector space and $0 \notin \Phi$.
(2) If $\alpha \in \Phi$, then $r\alpha \in \Phi$ for $r \in \mathbb{R}$, if and only if $r \in \{\pm 1\}$.
(3) For $\alpha \in \Phi$ let s_α denote the reflection on the hyperspace orthogonal to α:

$$s_\alpha(v) = v - \frac{2(v,\alpha)}{(\alpha,\alpha)}\alpha, \quad v \in V.$$

Then $s_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$.
(4) $2(\beta,\alpha)/(\alpha,\alpha) \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$.

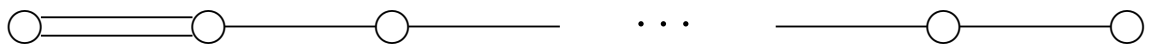
1.1.6 Weyl group and Dynkin diagram

Let Φ be a root system in the inner product space V. The group

$$W := W(\Phi) := \langle s_\alpha \mid \alpha \in \Phi \rangle \leq O(V)$$

is called the *Weyl group* of Φ. Another important notion is that of a *base* of Φ. This is a subset $\Pi \subset \Phi$ such that
(1) Π is a basis of V.
(2) Every $\alpha \in \Phi$ is an integer linear combination of Π with either only non-negative or only non-positive coefficients.

The Weyl group acts regularly on the set of bases of Φ. The *Dynkin diagram* of Φ is defined with respect to one such base. It is the graph with nodes $\alpha \in \Pi$, and $4(\alpha,\beta)^2/(\alpha,\alpha)(\beta,\beta)$ edges between the nodes α and β. For example, the Dynkin diagram of a root system of type B_r looks as follows.

B_r:


1.1.7 Chevalley groups

Chevalley groups are subgroups of automorphism groups of finite classical Lie algebras. A *Classical Lie algebra* is a Lie algebra corresponding to a finite-dimensional simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$.

These have been classified by Killing and Cartan in the 1890s in terms of root systems. Let Φ be the root system of $\mathfrak{g}$, and let Π be a base of Φ. It was shown by Chevalley, that $\mathfrak{g}$ has a particular basis, now called *Chevalley basis*, $\mathcal{C} = \{e_r \mid r \in \Phi, h_r, r \in \Pi\}$, such that all structure constants with respect to $\mathcal{C}$ are integers.

Let $\mathfrak{g}_{\mathbb{Z}}$ denote the $\mathbb{Z}$-form of $\mathfrak{g}$ constructed from $\mathcal{C}$, i.e. the set of $\mathbb{Z}$-linear combinations of $\mathcal{C}$ inside $\mathfrak{g}$. Then $\mathfrak{g}_{\mathbb{Z}}$ is a Lie algebra over the integers, free and of finite rank as an abelian group. If k is any field, then $\mathfrak{g}_k := k \otimes_{\mathbb{Z}} \mathfrak{g}_{\mathbb{Z}}$ is the *classical Lie algebra corresponding to* $\mathfrak{g}$.

1.1.8 Chevalley's construction (1955, [11])

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$ with Chevalley basis $\mathcal{C}$. For $r \in \Phi$, $\zeta \in \mathbb{C}$, there is $x_r(\zeta) \in \mathrm{Aut}(\mathfrak{g})$ defined by

$$x_r(\zeta) := \exp(\zeta \cdot \mathrm{ad}\; e_r).$$

Here, ad e_r denotes the endomorphism $x \mapsto [x, e_r]$ of $\mathfrak{g}$. The matrices of $x_r(\zeta)$ with respect to $\mathcal{C}$ have entries in $\mathbb{Z}[\zeta]$. This allows to define $x_r(t) \in \mathrm{Aut}(\mathfrak{g}_k)$ by replacing ζ by $t \in k$. Then

$$G := \langle x_r(t) \mid r \in \Phi, t \in k \rangle \leq \mathrm{Aut}(\mathfrak{g}_k)$$

is the *Chevalley group* corresponding to $\mathfrak{g}$ over k.

Names such as $A_r(q)$, $B_r(q)$, $G_2(q)$, $E_6(q)$, etc. refer to the type of the root system Φ of $\mathfrak{g}$.

1.1.9 Twisted groups (Tits, Steinberg, Ree, 1957–61)

Chevalley's construction gives many of the finite groups of Lie type, but not all. For example, the unitary group $\mathrm{GU}_n(q)$ is not a Chevalley group in this sense. However, $\mathrm{GU}_n(q)$ is obtained from the Chevalley group $\mathrm{GL}_n(q^2)$ by *twisting*:

Let σ denote the automorphism $(a_{ij}) \mapsto (a_{ij}^q)^{-tr}$ of $\mathrm{GL}_n(q^2)$. Then

$$\mathrm{GU}_n(q) = \mathrm{GL}_n(q^2)^\sigma := \{g \in \mathrm{GL}_n(q^2) \mid \sigma(g) = g\}.$$

Similar constructions give the twisted groups ${}^2E_6(q)$, ${}^3D_4(q)$, and the Suzuki and Ree groups ${}^2B_2(2^{2m+1})$, ${}^2G_2(3^{2m+1})$, ${}^2F_4(2^{2m+1})$. These constructions were found by Tits, Steinberg and Ree between 1957 and 1961 (see [80, 75, 70, 71]), although ${}^2B_2(2^{2m+1})$ was discovered in 1960 by Suzuki [79] by a different method.

1.2 Finite reductive groups

The construction discussed in this subsection introduces a decisive class of finite groups of Lie type.

1.2.1 Linear algebraic groups

Let $\bar{\mathbb{F}}_p$ denote the algebraic closure of the finite field $\mathbb{F}_p$. For the purpose of this survey, a *(linear) algebraic group* $\mathbf{G}$ over $\bar{\mathbb{F}}_p$ is a closed subgroup of $\mathrm{GL}_n(\bar{\mathbb{F}}_p)$ for some n. Here, and in the following, topological notions such as closedness refer to the *Zariski topology* of $\mathrm{GL}_n(\bar{\mathbb{F}}_p)$. The closed sets in the Zariski topology are the zero sets of systems of polynomial equations.

Example 1.5 (1) $\mathrm{SL}_n(\bar{\mathbb{F}}_p) = \{g \in \mathrm{GL}_n(\bar{\mathbb{F}}_p) \mid \det(g) = 1\}$.
(2) $\mathrm{SO}_n(\bar{\mathbb{F}}_p) = \{g \in \mathrm{SL}_n(\bar{\mathbb{F}}_p) \mid g^{tr} J g = J\}$ $(n = 2m + 1$ odd$)$.

The algebraic group $\mathbf{G}$ is *semisimple*, if it has no closed connected soluble normal subgroup $\neq 1$. It is *reductive*, if it has no closed connected unipotent normal subgroup $\neq 1$. In particular, semisimple algebraic groups are reductive. For a thorough treatment of linear algebraic group see the textbook by Humphreys [49].

1.2.2 Frobenius maps

Let $\mathbf{G} \leq \mathrm{GL}_n(\bar{\mathbb{F}}_p)$ be a connected reductive algebraic group. A *standard Frobenius map* of $\mathbf{G}$ is a homomorphism

$$F := F_q : \mathbf{G} \to \mathbf{G}$$

of the form $F_q((a_{ij})) = (a_{ij}^q)$ for some power q of p. (This implicitly assumes that $(a_{ij}^q) \in \mathbf{G}$ for all $(a_{ij}) \in \mathbf{G}$.)

Example 1.6 $\mathrm{SL}_n(\bar{\mathbb{F}}_p)$ and $\mathrm{SO}_{2m+1}(\bar{\mathbb{F}}_p)$ admit standard Frobenius maps F_q for all powers q of p.

A *Frobenius map* $F : \mathbf{G} \to \mathbf{G}$ is a homomorphism such that F^m is a standard Frobenius map for some $m \in \mathbb{N}$. If F is a Frobenius map, let $q \in \mathbb{R}$, $q \geq 0$ be such that q^m is a power of p with $F^m = F_{q^m}$.

1.2.3 Finite reductive groups

Let $\mathbf{G}$ be a connected reductive algebraic group over $\bar{\mathbb{F}}_p$ and let F be a Frobenius map of $\mathbf{G}$. Then

$$\mathbf{G}^F := \{g \in \mathbf{G} \mid F(g) = g\}$$

is a finite group. The pair $(\mathbf{G}, F)$ or the finite group $G := \mathbf{G}^F$ is called a *finite reductive group* or *finite group of Lie type*, though the latter terminology is also used in a broader sense.

Example 1.7 Let q be a power of p and let $F = F_q$ be the corresponding standard Frobenius map of $\mathrm{GL}_n(\bar{\mathbb{F}}_p)$, $(a_{ij}) \mapsto (a_{ij}^q)$. Then $\mathrm{GL}_n(\bar{\mathbb{F}}_p)^F = \mathrm{GL}_n(q)$, $\mathrm{SL}_n(\bar{\mathbb{F}}_p)^F = \mathrm{SL}_n(q)$, $\mathrm{SO}_{2m+1}(\bar{\mathbb{F}}_p)^F = \mathrm{SO}_{2m+1}(q)$.

All groups of Lie type, except the Suzuki and Ree groups can be obtained in this way by a **standard** Frobenius map. The projective special linear group $\mathrm{PSL}_n(q)$ is not a finite reductive group unless n and $q-1$ are coprime (in which case it is equal to $\mathrm{SL}_n(q)$).

For the remainder of this section, $(\mathbf{G}, F)$ denotes a finite reductive group over $\bar{\mathbb{F}}_p$.

1.2.4 The Lang–Steinberg theorem

One of the most important general results for a finite reductive group is the following theorem due to Lang and Steinberg.

Theorem 1.8 (Lang–Steinberg, 1956 [60]/1968 [78]) *If $\mathbf{G}$ is connected, the map $\mathbf{G} \to \mathbf{G}$, $g \mapsto g^{-1}F(g)$ is surjective.*

The assumption that $\mathbf{G}$ is connected is crucial here.

Example 1.9 Let $\mathbf{G} = \mathrm{GL}_2(\bar{\mathbb{F}}_p)$, and $F : (q_{ij}) \mapsto (a_{ij}^q)$, where q is a power of p. Then there exists $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{G}$ such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} a^q & b^q \\ c^q & d^q \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Rewriting this, we obtain the equation

$$\begin{bmatrix} a^q & b^q \\ c^q & d^q \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}.$$

Thus the Lang–Steinberg theorem asserts in this case that there is a solution to the system of equations:

$$a^q = b, \quad b^q = a, \quad c^q = d, \quad d^q = c, \quad ad - bc \neq 0.$$

The Lang–Steinberg theorem is used to derive structural properties of $\mathbf{G}^F$.

1.2.5 Maximal tori and the Weyl group

A *torus* of $\mathbf{G}$ is a closed subgroup isomorphic to $\bar{\mathbb{F}}_p^* \times \cdots \times \bar{\mathbb{F}}_p^*$. A torus is *maximal*, if it is not contained in any larger torus of $\mathbf{G}$. It is a crucial fact that any two maximal tori of $\mathbf{G}$ are conjugate. This shows that the following notion is well defined.

Definition 1.10 The Weyl group W of $\mathbf{G}$ is defined by $W := N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$, where $\mathbf{T}$ is a maximal torus of $\mathbf{G}$.

Example 1.11 (1) Let $\mathbf{G} = \mathrm{GL}_n(\bar{\mathbb{F}}_p)$ and $\mathbf{T}$ the group of diagonal matrices. Then $\mathbf{T}$ is a maximal torus of $\mathbf{G}$, $N_{\mathbf{G}}(\mathbf{T})$ is the group of monomial matrices, and $W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ can be identified with the group of permutation matrices, i.e. $W \cong S_n$.

(2) Next let $\mathbf{G} = \mathrm{SO}_{2m+1}(\bar{\mathbb{F}}_p)$ as defined in Example 1.2. Then

$$\mathbf{T} := \{\mathrm{diag}[t_1, \ldots, t_m, 1, t_m^{-1}, \ldots, t_1^{-1}] \mid t_i \in \bar{\mathbb{F}}_p^*, 1 \leq i \leq m\}$$

is a maximal torus of $\mathbf{G}$.

For $1 \leq i \leq m-1$ let $\dot{s}_i$ be the permutation matrix corresponding to the double transposition $(i, i+1)(m-i, m-i+1)$. Put

$$\dot{s}_m := \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix},$$

where I denotes the identity matrix of degree $m-1$. Then $\dot{s}_1, \ldots, \dot{s}_m$ are elements of $N_{\mathbf{G}}(\mathbf{T})$, and the cosets $s_i := \dot{s}_i\mathbf{T} \in W$, $1 \leq i \leq m$, generate W, which is thus a Coxeter group of type B_m (see below).

1.2.6 Maximal tori of finite reductive groups

A *maximal torus* of $(\mathbf{G}, F)$ is a finite reductive group $(\mathbf{T}, F)$, where $\mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$. A *maximal torus* of $G = \mathbf{G}^F$ is a subgroup T of the form $T = \mathbf{T}^F$ for some maximal torus $(\mathbf{T}, F)$ of $(\mathbf{G}, F)$.

Example 1.12 A *Singer cycle* in $\mathrm{GL}_n(q)$ is an irreducible cyclic subgroup of $\mathrm{GL}_n(q)$ of order $q^n - 1$. We will show below that a Singer cycle is a maximal torus of $\mathrm{GL}_n(q)$.

The maximal tori of $(\mathbf{G}, F)$ are classified (up to conjugation in G) by *F-conjugacy classes* of W. These are the orbits in W under the action $v.w := vwF(v)^{-1}$, $v, w \in W$.

1.2.7 The classification of maximal tori

Let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{G}$, $W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$.

Let $w \in W$, and $\dot{w} \in N_{\mathbf{G}}(\mathbf{T})$ with $w = \dot{w}\mathbf{T}$. By the Lang-Steinberg theorem, there is $g \in \mathbf{G}$ such that $\dot{w} = g^{-1}F(g)$. One checks that ${}^g\mathbf{T}$ is F-stable, and so $({}^g\mathbf{T}, F)$ is a maximal torus of $(\mathbf{G}, F)$. (Indeed, $F({}^g\mathbf{T}) = F(g)F(\mathbf{T})F(g)^{-1} = g(\dot{w}\mathbf{T}\dot{w}^{-1})g^{-1} = {}^g\mathbf{T}$ since $\dot{w} \in N_{\mathbf{G}}(\mathbf{T})$.)

The map $w \mapsto ({}^g\mathbf{T}, F)$ induces a bijection between the set of F-conjugacy classes of W and the set of G-conjugacy classes of maximal tori of $(\mathbf{G}, F)$. For more details see [10, Section 3.3].

We say that ${}^g\mathbf{T}$ is obtained from $\mathbf{T}$ by *twisting with w*.

1.2.8 The maximal tori of $\mathrm{GL}_n(q)$

Let $\mathbf{G} = \mathrm{GL}_n(\bar{\mathbb{F}}_p)$ and $F = F_q$ a standard Frobenius morphism, where q is a power of p.

Then F acts trivially on $W = S_n$, i.e. the maximal tori of $G = \mathrm{GL}_n(q)$ are parametrised by partitions of n. If $\lambda = (\lambda_1, \ldots, \lambda_l)$ is a partition of n, we write T_λ for the corresponding maximal torus. We have

$$|T_\lambda| = (q^{\lambda_1} - 1)(q^{\lambda_2} - 1) \cdots (q^{\lambda_l} - 1).$$

Each factor $q^{\lambda_i} - 1$ of $|T_\lambda|$ corresponds to a cyclic direct factor of T_λ of this order. This follows from the considerations in the next subsection.

1.2.9 The structure of the maximal tori

Let $\mathbf{T}'$ be an F-stable maximal torus of $\mathbf{G}$, obtained by twisting the reference torus $\mathbf{T}$ with $w = \dot{w}\mathbf{T} \in W$. This means that there is $g \in \mathbf{G}$ with $g^{-1}F(g) = \dot{w}$ and $\mathbf{T}' = {}^g\mathbf{T}$. Then

$$T' = (\mathbf{T}')^F \cong \mathbf{T}^{wF} := \{t \in \mathbf{T} \mid t = \dot{w}F(t)\dot{w}^{-1}\}.$$

Indeed, for $t \in \mathbf{T}$ we have $gtg^{-1} = F(gtg^{-1})$ $[= F(g)F(t)F(g)^{-1}]$ if and only if $t \in \mathbf{T}^{wF}$.

Example 1.13 Let $\mathbf{G} = \mathrm{GL}_n(\bar{\mathbb{F}}_p)$, and $\mathbf{T}$ the group of diagonal matrices. Let $w = (1, 2, \ldots, n)$ be an n-cycle. Then

$$\mathbf{T}^{wF} = \{\mathrm{diag}[t, t^q, \ldots, t^{q^{n-1}}] \mid t \in \bar{\mathbb{F}}_p, t^{q^n - 1} = 1\},$$

and so $\mathbf{T}^{wF}$ is cyclic of order $q^n - 1$. It also follows that the maximal torus of G corresponding to w acts irreducibly on $\mathbb{F}_q^n$ and thus is a Singer cycle. On the other hand, a maximal torus of G corresponding to an element of W not conjugate to w acts reducibly on V since it lies in a proper Levi subgroup. Since every semisimple element of G, in particular a generator of a Singer cycle, lies in some maximal torus of G, it follows that a Singer cycle is indeed a maximal torus.

1.3 BN-pairs

The following axiom system was introduced by Jacques Tits to allow a uniform treatment of groups of Lie type, not necessarily finite ones.

1.3.1 BN-pairs

We begin by defining what it means that a group has a BN-pair.

Definition 1.14 Let G be a group. The subgroups B and N of the group G form a BN-pair, if the following axioms are satisfied:

(1) $G = \langle B, N \rangle$;

(2) $T := B \cap N$ is normal in N;
(3) $W := N/T$ is generated by a set S of involutions;
(4) If $\dot{s} \in N$ maps to $s \in S$ (under $N \to W$), then $\dot{s}B\dot{s} \neq B$;
(5) For each $n \in N$ and $\dot{s}$ as above, $(B\dot{s}B)(BnB) \subseteq B\dot{s}nB \cup BnB$.

The group $W = N/T$ is called the *Weyl group* of the BN-pair of G. It is a Coxeter group with Coxeter generators S.

1.3.2 Coxeter groups

Let $M = (m_{ij})_{1 \leq i,j \leq r}$ be a symmetric matrix with $m_{ij} \in \mathbb{Z} \cup \{\infty\}$ satisfying $m_{ii} = 1$ and $m_{ij} > 1$ for $i \neq j$. The group

$$W := W(M) := \left\langle s_1, \ldots, s_r \mid (s_i s_j)^{m_{ij}} = 1 (i \neq j), s_i^2 = 1 \right\rangle_{\text{group}}$$

(where the relation $(s_i s_j)^{m_{ij}} = 1$ is omitted if $m_{ij} = \infty$), is called the *Coxeter group* of M, the elements $s_1, \ldots, s_r$ are the *Coxeter generators* of W.

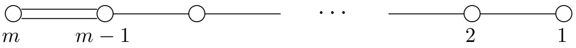
The relations $(s_i s_j)^{m_{ij}} = 1$ $(i \neq j)$ are called the *braid relations*. In view of $s_i^2 = 1$, they can be written as

$$s_i s_j s_i \cdots = s_j s_i s_j \cdots \quad m_{ij} \text{ factors on each side.}$$

The matrix M is usually encoded in a *Coxeter diagram*, a graph with nodes corresponding to $1, \ldots, r$, and with the number of edges between nodes $i \neq j$ equal to $m_{ij} - 2$.

Example 1.15 The involutions s_i introduced in Example 1.11(2) satisfy the relations $s_i^2 = 1$ for $1 \leq i \leq m$, $(s_i s_{i+1})^3 = 1$ for $1 \leq i \leq m-1$ and $(s_{m-1}s_m)^4 = 1$. All other pairs of the s_i commute. The matrix encoding these relations is called a Coxeter matrix of type B_m. Its Coxeter diagram is as follows.

B_m:



1.3.3 The BN-pair of $\mathrm{GL}_n(k)$ and of $\mathrm{SO}_n(k)$

Let k be a field and $G = \mathrm{GL}_n(k)$. Then G has a BN-pair with:

- B the group of upper triangular matrices;
- N the group of monomial matrices;
- $T = B \cap N$ the group of diagonal matrices;
- $W = N/T \cong S_n$ the group of permutation matrices.

Let $n = 2m+1$ be odd and let $\mathrm{SO}_n(k) = \{g \in \mathrm{SL}_n(k) \mid g^{tr}Jg = J\} \leq \mathrm{GL}_n(k)$ be the orthogonal group. If B, N are as above for $\mathrm{GL}_n(k)$, then

$$B \cap \mathrm{SO}_n(k), N \cap \mathrm{SO}_n(k)$$

is a BN-pair of $\mathrm{SO}_n(k)$. (This would not have been the case had we defined $\mathrm{SO}_n(k)$ with respect to an orthonormal basis as $\mathrm{SO}_n(k) = \{g \in \mathrm{SL}_n(k) \mid g^{tr}g = I\}$.) Using Examples 1.11 and 1.15 we see that the Weyl group of $\mathrm{SO}_n(k)$ is a Coxeter group of type B_m.

1.3.4 Split BN-pairs of characteristic p

Let G be a group with a BN-pair (B, N). This is said to be a *split BN-pair of characteristic* p, if the following additional hypotheses are satisfied:

(6) $B = UT$ with $U = O_p(B)$, the largest normal p-subgroup of B, and T a complement of U.

(7) $\bigcap_{n \in N} nBn^{-1} = T$. (Recall $T = B \cap N$.)

Example 1.16 (1) A semisimple algebraic group $\mathbf{G}$ over $\bar{\mathbb{F}}_p$ and a finite group of Lie type of characteristic p have split BN-pairs of characteristic p.

In $\mathbf{G}$ one chooses a maximal torus $\mathbf{T}$ and a maximal closed connected soluble subgroup $\mathbf{B}$ of $\mathbf{G}$ containing $\mathbf{T}$. Such a $\mathbf{B}$ is called a *Borel subgroup* of $\mathbf{G}$. Then $\mathbf{B}$ and $N_{\mathbf{G}}(\mathbf{T})$ form a split BN-pair of $\mathbf{G}$ of characteristic p.

(2) If $G = \mathrm{GL}_n(\bar{\mathbb{F}}_p)$ or $\mathrm{GL}_n(q)$, q a power of p, then U is the group of upper triangular unipotent matrices. In the latter case, U is a Sylow p-subgroup of G.

1.3.5 Parabolic subgroups and Levi subgroups

Let G be a group with a split BN-pair of characteristic p. Any conjugate of B is called a *Borel subgroup* of G. A *parabolic subgroup* of G is one containing a Borel subgroup.

Let $P \leq G$ be a parabolic subgroup. Then

$$P = U_P L = L U_P \tag{1}$$

such that $U_P = O_p(P)$ is the largest normal p-subgroup of P, and L is a complement to U_P in P. The decomposition (1) is called a *Levi decomposition* of P, and L is a *Levi complement* of P, and a *Levi subgroup* of G.

A Levi subgroup is itself a group with a split BN-pair of characteristic p.

1.3.6 Examples for parabolic subgroups

In classical groups, parabolic subgroups are the stabilisers of isotropic subspaces. Let $G = \mathrm{GL}_n(q)$, and $(\lambda_1, \ldots, \lambda_l)$ a partition of n. Then

$$P = \left\{ \begin{bmatrix} \mathrm{GL}_{\lambda_1}(q) & \star & \star \\ & \ddots & \star \\ & & \mathrm{GL}_{\lambda_l}(q) \end{bmatrix} \right\}$$

is a typical parabolic subgroup of G. A corresponding Levi subgroup is

$$L = \left\{ \begin{bmatrix} \mathrm{GL}_{\lambda_1}(q) & & \\ & \ddots & \\ & & \mathrm{GL}_{\lambda_l}(q) \end{bmatrix} \right\} \cong GL_{\lambda_1}(q) \times \cdots \times \mathrm{GL}_{\lambda_l}(q).$$

If B denotes, once again, the group of upper triangular matrices in G, then a Levi decomposition of B is given by $B = UT$ with T the diagonal matrices and U the upper triangular unipotent matrices.