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Representation of rings over skew fields

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CAMBRIDGE UNIVERSITY PRESS
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CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521278539

First published 1985
Re-issued in this digitally printed version 2008

A catalogue record for this publication is available from the British Library

Library of Congress Catalogue Card Number: 84–22996

ISBN 978-0-521-27853-9 paperback

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Cambridge University Press
978-0-521-27853-9 - Representation of Rings over Skew Fields
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To Nicky

PREFACE

The finite dimensional representations of a ring over commutative fields have been studied in great detail for many types of ring, for example, group rings or the enveloping algebras of finite dimensional Lie algebras, but little is known about the finite dimensional representations of a ring over skew fields although such information might be of great use. The first part of this book is devoted to a classification of all possible finite dimensional representations of an arbitrary ring over skew fields in terms of simple linear data on the category of finitely presented modules over the ring. The second part is devoted to a fairly detailed study of those skew fields that arise in the first part and in the work of Cohn on firs and skew fields.

As has been said, the main goal at the beginning is to study finite dimensional representations of a ring over skew fields. An alternative view of this is that we should like to classify all possible homomorphisms from a ring to simple artinian rings; such a study was carried out in the case of one dimensional representations which are simply homomorphisms to skew fields by Cohn who showed that these homomorphisms are determined by which sets of matrices become zero-divisors over the skew field and gave a characterisation of the sets of matrices that could be exactly those that become singular under a homomorphism to a skew field. This theory has a particular application to firs, rings such that every left and right ideal are free of unique rank to show that they have universal homomorphisms to skew fields. This applies to the free algebra over a commutative field, and the ring coproduct of a family of skew fields amalgamating a common skew subfield, and gives the free skew field on a generating set and the skew field coproduct with amalgamation.

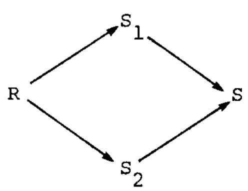
In order to classify homomorphisms from a ring to simple artinian rings, it is necessary to investigate what type of information a homomorphism gives on the ring. The most obvious point is that it induces certain rank functions on the modules over the ring; over a simple artinian ring, $S = M_n(D)$, where D is a skew field, every module is a direct sum of copies of the simple module, and the free module of rank one is the direct sum of n copies of the simple module, so we can assign a rank to the finitely generated modules over the ring taking values in $\frac{1}{n}\mathbb{Z}$ so that the free module on one generator has rank 1. If there is a homomorphism from R to S, we may assign ranks to the f.g. projectives or more generally the finitely presented modules over R by $\rho(M)$ is the rank over S of $M \otimes_R S$. Considering the rank functions induced on finitely generated projectives is important for constructing universal homomorphisms from an hereditary ring to simple artinian rings, whilst rank functions on finitely presented modules are precisely what is needed to classify all homomorphisms from the ring to simple artinian rings. The main result states that a rank function ρ on the finitely presented modules over a k-algebra R taking values in $\frac{1}{n}\mathbb{Z}$ arises from a homomorphism to a simple artinian ring if and only if it satisfies the following axioms:

1. $\rho(R^1) = 1$;
2. $\rho(A \oplus B) = \rho(A) + \rho(B)$;
3. if $A \to B \to C \to 0$ is an exact sequence of finitely presented modules then $\rho(C) \le \rho(B) \le \rho(A) + \rho(C)$.

If R is a ring that is not a k-algebra, it is necessary to have a fourth axiom:

4. $\rho(R/mR) = 0$ or 1 for any integer m.

Two homomorphisms $\alpha_i : R \to S_i$ induce the same rank function if and only if there is a commutative diagram of rings:



In chapter 1, we begin the study of hereditary rings and rank functions on finitely generated projectives over them. In the main, it is a study of the category of finitely generated projectives and the ranks that the rank function induces on the maps in the category. It is shown that this behaves in a very similar way to the rank functions on von Neumann regular rings, which is where the notion of a rank function came from; this analogy is developed to its logical conclusion in chapter 6, where it is shown that a rank function taking values in the real numbers defined on the f.g. projectives over an hereditary ring must arise from a homomorphism to a von Neumann regular ring.

Chapter 2 sets forth the first of the ring constructions that are needed in order to construct homomorphisms, the ring coproduct amalgamating a semisimple artinian subring. On the whole, it is a summary without proofs of Bergman's coproduct theorems. Chapter 3 shows how projective rank functions behave under the coproduct construction. It is also shown that if a module M over R_1 requires n generators then the module $M \otimes_R R'$ over the ring coproduct R' of R_1 and R_2 amalgamating a skew subfield F still requires n generators provided that there are finitely generated modules over R_2 requiring arbitrary large numbers of generators; the condition is clearly necessary. This may be regarded as the analogue of the Grushko Neumann theorem. The results on projective rank functions are applied to prove a recent theorem due to Linnell; a finitely generated group is accessible if there is a bound on the size of finite subgroups.

Chapter 4 presents the second important construction, adjoining universal inverses to maps between finitely generated projectives over a ring; this was studied by Cohn for matrices in order to construct homomorphisms to skew fields, but it has usually been regarded as a difficult technique, although it has arisen, usually in disguised form, in a number of contexts. For example, one of the methods used for showing that some finite dimensional algebra is of wild representation type amounts to adjoining a universal inverse to a suitable map. There are a number of ways of studying this construction developed recently which make it a little easier to calculate with and to think about, and the aim of this chapter is to present them. At the end, the algebraic K-theory of a universal localisation is discussed; there is an exact sequence for the algebraic K-theory that generalises the Bass, Murthy sequence for central localisation.

Chapter 5 pulls together the various pieces presented in the first four chapters in order to construct universal homomorphisms from an hereditary algebra with a rank function on its finitely generated projectives to a simple artinian ring. The idea is fairly simple; given a rank function ρ on an hereditary ring R, we ask which maps between finitely generated projectives have a chance of becoming invertible under a homomorphism from R to a simple artinian ring that induces the given rank function; if $\alpha: P \to Q$ is such a map, then $\rho(P) = \rho(Q)$ and α cannot factor through a projective of smaller rank. Such maps are called full maps. The universal localisation of an hereditary ring at all full maps with respect to a rank function taking values in $\frac{1}{n}\mathbb{Z}$ is a perfect hereditary ring and it is simple artinian in a large number of interesting cases. Chapter 6 completes this circle of ideas by showing that if R is an hereditary ring with a rank function on its finitely generated projectives taking values in the real numbers, there is a homomorphism from R to a von Neumann regular ring with a unique rank function that induces this rank function on R. This theorem actually holds provided that all countably generated right and left ideals over R are projective, which means that it applies to a von Neumann regular ring with a rank function.

Chapter 7 contains a number of results on homomorphisms to simple artinian rings beyond those that were discussed above. The space of all possible rank functions on finitely presented modules over a k-algebra that satisfy the axioms we stated earlier form in a natural way a $\mathbb{Q}$-convex subset of an infinite dimensional vector space. Given two rank functions that satisfy the axioms given, so does the rank function $q_1\rho_1 + q_2\rho_2$ where q_1 and q_2 are positive rationals such that $q_1 + q_2 = 1$. It is shown in the course of chapter 7 that every rank function ρ has a unique expression in the form $\Sigma q_i\rho_i$ where q_i are positive rationals such that $q_i = 1$ and ρ_i are rank functions that cannot be written as the weighted sum of different rank functions. So, the space of all possible rank functions is a sort of locally finite dimensional $\mathbb{Q}$-simplex.

The methods and theorems developed in the first part of this book are of great use in studying the skew fields constructed by Cohn, and the second half of this book is a fairly detailed investigation along these lines.

In chapter 8, we investigate what is known about the centre of the skew field and simple artinian coproduct. We have a complete answer when we amalgamate over a central subfield; however, the results are rather incomplete for simple artinian coproducts where none of the factors are skew fields. Chapter 9 continues with a detailed discussion of the finite dimensional division subalgebras of skew field coproducts and a number of other related skew fields. As an example of the odd results that occur, it is shown that if E_1 and E_2 are skew fields containing no elements algebraic over the central subfield k, then $E_1 \circ_k E_2$ the skew field coproduct of E_1 and E_2 amalgamating k can sometimes contain a finite dimensional field extension L of K, but if it does, $[L:k]$ must be divisible by two

Cambridge University Press
978-0-521-27853-9 - Representation of Rings over Skew Fields
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different primes; there is an example where $[L:k] = 6$. There is also an example of a skew field D with centre k such that $D\otimes_k k_s$ and $D\otimes_k kP^{-\infty}$ are skew fields where k_s is the separable closure of k, and $kP^{-\infty}$ is the inseparable closure of k but $D\otimes_k \bar{k} \cong M_p(D')$ for some skew field D', where $\bar{k}$ is the algebraic closure of k; this settles a question of Cohn and Dicks.

Chapter 10 develops the technique of the universal bimodule of derivations in order to distinguish between various non-isomorphic skew fields. In particular, it is shown that the free skew field on m generators cannot be isomorphic to the free skew field on n generators for $m \neq n$. It also gives a way for recognising when a skew field is a universal localisation of an hereditary subring.

Chapter 11 continues the investigation of the skew subfields of a skew field coproduct; we are particularly interested in the commutative subfields of such skew fields and in centralisers in matrix rings over a free skew field. In the first case, it is possible to bound the transcendence degree of commutative subfields of a skew field coproduct in terms of the transcendence degree of commutative subfields of the factors and the amalgamated skew field of the coproduct. For centralisers, it is shown that a skew subfield D with transcendental centre of $M_n(F)$ where F is a free skew field over k has a finitely generated centre over k of transcendence degree 1, its dimension over its centre is finite, and this dimension must divide n^2. At the end of the chapter, it is shown that a 2 generator skew subfield of a free skew field must either be free on those 2 generators or else it is commutative.

Chapter 12 develops the characterisation of the universal localisations of hereditary rings that are skew fields which was developed in chapter 10 into a characterisation of simple artinian universal localisations of hereditary rings; then it is shown that if T is a subring of a simple artinian universal localisation of the hereditary ring R that contains the image of R so that the map from R to T is an epimorphism, then T is itself a universal localisation of R. It follows from this result that epic endomorphisms of the free algebra over a commutative field are isomorphisms; this is the non-commutative analogue of the Jacobian conjecture.

The final chapter presents among other things a solution to an old problem; it is shown that for any pair of integers $a,b > 1$, there exists an extension of skew fields $E \supset F$ such that the left dimension of

E over F is a, whilst the right dimension is b. By extending the construction, it is possible to construct a new class of hereditary artinian rings of finite representation type. In order to effect these constructions, we develop a new type of hereditary ring construction, the bimodule amalgam rings; these are rings generated by two simple artinian rings S and S' subject only to conditions on the S, S' bimodule $SS' = \{\Sigma s_i s_i' : s_i \in S, s_i' \in S'\}$. When we are able to show that these hereditary rings have a rank function, their properties are of particular interest. In addition to the results mentioned above, they also allow us to construct isomorphisms between skew fields that at first glance appear to be quite different. As an example, it is shown that if E_1 and E_2 are division subalgebras of the skew field F such that $[E_1:k] = [E_2:k]$ where k is a central subfield, then $F \circ_k E_1$ is isomorphic to $F \circ_k E_2$.

There are a number of people that I should like to thank for their encouragement and help during the proving of these results and subsequently during the time that I was writing them down. The first person I should like to thank is Warren Dicks with whom I have discussed most of the results of this book; his care and accuracy have been of great assistance to me and many of the results have arisen out of conversations between us. I should also like to thank Paul Cohn for his interest and encouragement; I owe him a particular debt for having proven the first results in this area. I should also like to thank Rufus Neal for bearing with me despite the length of time that it has taken me to get this book into his hands at Cambridge University Press, and I am very grateful to Diane Quarrie for typing this book so well from a partial typescript of poor quality.