PART I

Homomorphisms to simple artinian rings

1 HEREDITARY RINGS AND PROJECTIVE RANK FUNCTIONS

Definitions and preliminaries

In this chapter, we introduce the two main subjects of the first part of this book; hereditary rings and the projective rank functions on the rings, which we need in order to study their homomorphisms to simple artinian rings.

A left hereditary ring is one such that all left ideals are projective modules. We shall be interested in a number of variants of this definition; a left semihereditary ring is one such that all finitely generated left ideals are projective and a left $\aleph_0$-hereditary ring is one such that all countably generated left ideals are projective. We shall often need to consider the two-sided properties, whose definitions we leave to the reader; our results tend to work most often in the case of two-sided $\aleph_0$-hereditary rings. We shall tend to miss out the words 'two-sided', when using these conditions. There is a two-sided condition implied by all of the one-sided conditions above; a ring is weakly semihereditary if, for all pairs of maps $\alpha : P_0 \to P_1, \beta : P_1 \to P_2$ between finitely generated projective modules such that $\alpha\beta = 0$, then $P_1 = P_1' \oplus P_1''$, where the image of α lies in P_1' and the kernel of β contains P_1'. This is a two-sided condition because of the duality between the category of finitely generated left projectives and finitely generated right projectives induced by $\mathrm{Hom}_R(\_,R)$.

We shall abbreviate 'finitely generated' to f.g., 'finitely presented' to f.p., and, in the case of vector spaces over a skew field, we abbreviate 'finite dimensional' to f.d.

Lemma 1.1 A left semihereditary ring is weakly semihereditary.

Proof: Suppose that $\alpha : P_0 \to P_1, \beta : P_1 \to P_2$ are two maps such that $\alpha\beta = 0$, where P_i is a finitely generated projective for $i = 0, 1, 2$. The image of β is a projective module, so $P_1 = \mathrm{im}\beta \oplus \ker\beta$, and the image of α lies in the kernel of β.

A good reason for introducing the notion of a weakly semihereditary ring is the following theorem due to Bergman.

Theorem 1.2 Every projective module over a weakly semihereditary ring is a direct sum of finitely generated projective modules.

We refer the reader to 0.2.9 of Cohn (71) for a proof of this result.

We shall work as long as possible with weakly semihereditary rings; however, we shall eventually be forced to restrict our attention to two-sided $\aleph_0$-hereditary rings. This class of rings draws much of its initial interest from the fact that all von Neumann regular rings have this property. By a von Neumann regular ring, we mean a ring R, such that for all x in R, there exists an element y such that $xyx = x$; we shall see that there are interesting connections between these classes of rings.

Much of the work of this chapter is just a study of the category of finitely generated left projective modules over a weakly semihereditary ring. This has been done with a great deal of success for semifirs and firs by Cohn (71); a fir is a ring such that all left ideals and right ideals are free of unique rank, and a semifir is a ring such that all finitely generated left ideals (and so, all such right ideals too) are free of unique rank. In this case, the arguments work well because we have a good notion of the size of a finitely generated projective, and so we would like to have a generalisation of this idea for other rings. The relevant idea comes from the theory of von Neumann regular rings.

Given a ring, R, we associate to it the abelian monoid $P_{\oplus}(R)$ of isomorphism classes of f.g. projectives under direct sum. We may also associate to it a pre-ordered abelian group, the Grothendieck group, $K_0(R)$. It is generated by the isomorphism classes of finitely generated left projective modules [P], subject to the relations $[P\oplus Q] = [P] + [Q]$, for every pair of isomorphism classes [P], [Q]. The pre-order is given by specifying a positive cone, by which we mean simply a distinguished additive submonoid of positive elements, and, in this case we take the isomorphism classes of finitely generated projective modules, [P]. It is clear that $K_0(R)$ is the universal group associated to P(R). Two projectives P and Q are said to be stably isomorphic when $[P] = [Q]$; this is equivalent to the existence of an equation $P \oplus R^n \cong Q \oplus R^n$.

A projective rank function on a ring R is a homomorphism of pre-ordered groups, $\rho : K_0(R) \to \mathbb{R}$, the real numbers, such that $\rho([R^1]) = 1$. By definition, $\rho([P]) \geq 0$; we shall call a rank function faithful if $\rho([P]) > 0$, for all non-zero P. We shall often simplify the notation by writing $\rho(P)$ for $\rho([P])$. We note that a projective rank function is a left, right dual notion, because of the duality $\mathrm{Hom}_R(-,R)$.

A partial projective rank function is a homomorphism of pre-ordered groups $\rho : A \to \mathbb{R}$, where A is a subgroup of $K_0(R)$, containing $[R^1]$, and the partial order is that induced from $K_0(R)$ by restriction.

We recall theorem 18.1 of Goodearl (79):

Theorem 1.3 Every partial projective rank function extends to a projective rank function on R.

This result allows us to characterise those rings that have a projective rank function. We say that a ring has unbounded generating number if for every natural number n, there is a finitely generated module, M, requiring at least n generators. It is an easy check that this equivalent to the condition that for no m is there an equation of the form $R^m \cong R^{(m+1)} \oplus P$; and this is a left, right dual condition, which justifies the two-sided nature of our definition. Cohn mentions this class of rings in (Cohn 71) under the guise of rings such that for all n, the n by n identity matrix cannot be written as an n by $(n-1)$ matrix times an $(n-1)$ by n matrix. We leave it to the reader to check the equivalence.

Theorem 1.4 A ring has a projective rank function, if and only if it has unbounded generating number.

Proof: Certainly, if R has a projective rank function, it must have unbounded generating number.

Conversely, if R has unbounded generating number, the subgroup of $K_0(R)$ generated by $[R^1]$ is isomorphic to $\mathbb{Z}$, and under the isomorphism, no stably free projective module can have negative image. So this isomorphism defines a partial projective rank function on R, which must extend to a projective rank function by 1.3.

We have shown that most rings have a projective rank function; in

Cambridge University Press
978-0-521-27853-9 - Representation of Rings over Skew Fields
A. H. Schofield
Excerpt
More information

fact, projective rank functions arise quite naturally on rings and one is forced to study them in order to solve certain types of problems.

If S is a simple artinian ring, it has the form $M_n(D)$ for some skew field D, and so, $K_0(S)$ can be identified in a natural way with $\frac{1}{n}\mathbb{Z}$; so in this case, we have a unique rank function. If we have a homomorphism from a ring R to S, this induces a homomorphism from $K_0(R)$ to $K_0(S)$, which is naturally isomorphic to $\frac{1}{n}\mathbb{Z}$; therefore, homomorphisms to simple artinian rings induce rank functions to $\frac{1}{n}\mathbb{Z}$, and we shall need to consider such rank functions in order to study homomorphisms to simple artinian rings. More generally, many von Neumann regular rings have rank functions so that in order to study homomorphisms to von Neumann regular rings we shall need to consider quite general projective rank functions.

These projective rank functions appear naturally in the representation theory of finite dimensional algebras, for if R is a finite dimensional algebra over the field k, and M is a finite dimensional module, $[M:k] = m$, this defines a homomorphism from R to $M_m(k)$ and so determines a rank function ρ on R; it is easy to see that if P is a principal projective module over R, $P = Re$, then

$$\rho(P) = \frac{[Me:k]}{[M:k]}$$

which determines the projective rank function, since all f.g. projective modules are direct sums of principal projective modules for an artinian ring.

Another class of rings with a projective rank function that occurs naturally are the group rings in characteristic 0. We have a trace function on the group ring FG, where F is a field of characteristic 0 and G is a group given by $tr(\sum_i f_i g_i) = f_0$, where g_0 is the identity element of the group. We extend this to a trace function on the ring $M_n(FG)$ in the natural way and then we define the rank of an f.g. projective P to be the trace of an idempotent e in $M_n(FG)$ such that $FG^n e \cong P$. It is well known that this is well-defined, taking values in $\mathbb{Q}$, and that it is a faithful projective rank function. This will turn out to be useful to us later on in proving results due to Linnell on accessibility of f.g. groups.

Trace ideals

It is often useful to be able to work with a faithful rank function on a ring rather than one that is not; so we should like to have a way of getting rid of the projectives of rank zero. There is a standard way of dealing with this problem; we define the trace ideal of a set of f.g. projective modules, I, closed under direct sum to be the set of elements, T, that lie in the image of some map from one of these projectives to the free module of rank 1. It is easy to see that this set is an ideal, in fact, an idempotent ideal known as the trace ideal of the projectives in I, and that R/T is the universal R-ring such that $R/T \otimes_R P = 0$, for all P in I. We wish to study the behaviour of this construction.

Theorem 1.5 Let I be a set of f.g. projective modules closed under direct sum over a ring R and let T be the trace ideal of this set of projectives. For a f.g. projective, Q, $R/T \otimes_R \; = 0$, if and only if Q is a direct summand of an element of I. The monoid of induced projective modules over R/T is the quotient of $P_{\oplus}(R)$ by the relation $P \sim P'$, if and only if $P \oplus Q \cong P' \oplus Q'$, where Q and Q' are direct summands of elements of I.

Proof: Suppose that $R/T \otimes_R Q = 0$, then every element of Q lies in the image of a map from an element of I to Q; since Q is finitely generated, there must be a surjective map from an element of I to Q, which proves the first assertion.

Suppose that $R/T \otimes_R \alpha : R/T \otimes_R P \to R/T \otimes_R P'$ is an isomorphism over R/T; so there is a surjection: $\alpha \oplus \beta : P \oplus Q \to P'$, where Q is a direct summand of an element of I. Therefore, $0 \to \ker \alpha \oplus \beta \to P \oplus Q \to P' \to 0$ is a split exact sequence, where $R/T \otimes_R \alpha \oplus \beta$ is an isomorphism, since it equals $R/T \otimes_R \alpha$. So $\ker \alpha \oplus \beta$ becomes 0 over R/T, and must be a direct summand of an element of I. Therefore, as required, we have an equation of the form $P \oplus Q \cong P' \oplus Q'$. The converse is clear.

We have the following consequence:

Theorem 1.6 Let R be a ring with a projective rank function ρ; let T be the trace ideal of the projectives of rank 0; then ρ extends to a projective rank function on R/T.

Proof: By the last theorem, there is a partial projective rank function defined on the image of $K_0(R)$ in $K_0(R/T)$, induced by ρ. By theorem 1.3, this extends to a rank function on R/T.

We can do rather better than this on a weakly semihereditary ring. First, we need the following result on the behaviour of $P_{\oplus}$ on passing to the quotient by a trace ideal over a weakly semihereditary ring.

Theorem 1.7 Let R be a weakly semihereditary ring, and let T be the trace ideal of the set of f.g. projectives, I, closed under direct sum; then R/T is a weakly semihereditary ring, and $P_{\oplus}(R/T)$ is the quotient of $P_{\oplus}(R)$ by the relation $P \sim P'$ if and only if $P \oplus Q \cong P' \oplus Q'$, where Q and Q' are direct summands of an element of I.

Proof: We denote passage to R/T by bars, so $\bar{R} = R/T$.

Let $\bar{\alpha} : \bar{P} \to \bar{P}', \beta : \bar{P}' \to \bar{P}''$ be a pair of maps such that $\bar{\alpha}\bar{\beta} = 0$; then over R, $\alpha\beta = \gamma\delta, \gamma : P \to Q, \delta : Q \to P''$, where Q is an element of I. So, over R, we have

$$(\alpha \ \gamma) \begin{pmatrix} \beta \\ -\delta \end{pmatrix} = 0$$

and we note that $(\overline{\alpha\gamma}) = \bar{\alpha}$, and $\overline{\begin{pmatrix} \beta \\ -\delta \end{pmatrix}} = \bar{\beta}$.

Since R is weakly semihereditary, $P' \oplus Q \cong P_1 \oplus P_2$, where $\operatorname{im}(\alpha\gamma) \subseteq P_1$, and $\ker \begin{pmatrix} \beta \\ -\delta \end{pmatrix} \supseteq P_1$; therefore, $\bar{P}' \cong \bar{P}_1 \oplus \bar{P}_2$, where $\operatorname{im}\bar{\alpha} \subseteq \bar{P}_1 \subseteq \ker \bar{\beta}$, that is, the weakly semihereditary condition is satisfied for maps between induced projectives.

Let $e : R^n \to R^n$ be a map such that $\bar{e}^2 = \bar{e}$; that is, $\bar{e}(\overline{1-e}) = 0$. So, by the previous argument, $R^n \cong P_1 \oplus P_2$, where $\operatorname{im} \bar{e} \subseteq \bar{P}_1 \subseteq \ker(\overline{1-e})$; but the image of $\bar{e}$ is equal to the kernel of $(\overline{1-e})$, so $\operatorname{im} \bar{e} = \bar{P}_1$, which shows that all f.g. projectives are induced, and, in consequence, R/T is weakly semihereditary.

The rest follows from theorem 1.5.

This allows us to pass from a weakly semihereditary ring with a projective rank function to a weakly semihereditary ring with a faithful projective rank function, simply by killing the projectives of rank 0. We summarise this special case:

Theorem 1.8 Let R be a weakly semihereditary ring with a projective rank function ρ; let T be the trace ideal of the f.g. projectives of rank 0; then R/T is a weakly semihereditary ring, and ρ induces a faithful projective rank function on R/T. If ρ takes values in $\frac{1}{n}\mathbb{Z}$, then R/T is semihereditary on either side.

Proof: All is clear except for the last remark. In this case, R/T is a weakly semihereditary ring with a faithful projective rank function, taking values in $\frac{1}{n}\mathbb{Z}$. Let M be a finitely generated left ideal and let P be a f.g. projective over R/T of minimal rank such that there is a surjection $\alpha : P \to M$. If x lies in the kernel of this surjection, we have a sequence:

$$R/T \to P \to M \subseteq R,$$

whose composite is 0; so $P \cong P_1 \oplus P_2$, where x is in P_1, which is in the kernel of the surjection. Hence, $\alpha|P_2 : P_2 \to M$, is a surjection. Since $\rho(P_2) < \rho(P)$ unless $x = 0$, we deduce that $\alpha : P \to M$ must be an isomorphism.

We have already noted that trace ideals must be idempotent; curiously, the converse is true for left hereditary rings as we see next.

Theorem 1.9 Let R be a left hereditary ring, and let I be an idempotent ideal; then I is a trace ideal.

Proof: As a left module over R, I is projective and since $I = I^2$, $R/I \otimes_R I = I/I^2$. Hence, the trace ideal of the projective module I must contain I, but it can be no larger, since its image in R/I is trivial.

The inner projective rank

If we have a partial projective rank function, ρ_A, defined on the subgroup A of $K_0(R)$, we define the generating number with respect to ρ_A of a finitely generated left module M over R by the formula:

$$g.p_A(M) = \inf_{[P]}\{\rho_A(P): [P] \text{ is in } A, \exists \text{ a surjection } P \to M\} .$$

If all stably free modules are free of unique rank, and A' is the subgroup of $K_0(R)$ generated by $[R^1]$, the generating number with respect to $\rho_{A'}$, where $\rho_{A'}$ is the unique rank function defined on A is the minimal number of generators of a module. So we could hope and we shall show that our more general notion is a useful refinement.

We intend to use a projective rank function ρ to analyse the

Cambridge University Press
978-0-521-27853-9 - Representation of Rings over Skew Fields
A. H. Schofield
Excerpt
More information

category of f.g. projectives over a ring R. We have a rank associated to each object of the category, so, our next aim is to give each map a rank. Let $\alpha : P \to Q$ be a map between two f.g. projectives; we define the inner projective rank of the map with respect to ρ to be given by the formula:

$$\rho(\alpha) = \inf_{[P']}\{\rho(P') : \exists \text{ a commutative diagram } P \to Q \text{ via } P \to P' \to Q\}$$

It is sometimes useful to have a related notion to hand; we define the left nullity of α to be $\rho(P) - \rho(\alpha)$. Similarly, the right nullity of α is defined by $\rho(Q) - \rho(\alpha)$. $\rho(P) = \rho(Q)$, the nullity of α is $\rho(P) - \rho(\alpha)$.

We may relate the inner projective rank of a map and the generating number of suitable modules.

Lemma 1.10 Let R be a ring with a projective rank function ρ; then the inner rank of a map $\alpha : P \to Q$ is equal to the following: $\inf_M \{g.\rho(M) : \alpha(P) \subseteq M \subseteq Q$, where M is a f.g. submodule of $Q\}$.

In particular, if R is a left semihereditary ring,

$$\rho(\alpha) = \inf_{P'}\{\rho(P') : \alpha(P) \subseteq P' \subseteq Q\}.$$

Proof: If there is a commutative diagram $P \to Q$ via $P \to P' \to Q$, then $\alpha(P) \subseteq \beta(P') \subseteq Q$ and $g.\rho\ (\beta(P')) \leq \rho(P')$, so $\rho(\alpha) \geq \inf_M\{g.\rho(M) : \alpha(P) \subseteq M \subseteq Q\}$.

Conversely, if $\alpha(P) \subseteq M \subseteq Q$, and there exists a surjection $P' \to M$, we have a commutative diagram $P \to Q$ via $P \to P' \to Q$, since P is projective.

Hence, $\rho(\alpha) = \inf_M\{g.\rho(M) : \alpha(P) \subseteq M \subseteq Q\}$.

A map $\alpha : P \to Q$ is said to be left full with respect to ρ if $\rho(\alpha) = \rho(P)$, and it is right full if $\rho(\alpha) = \rho(Q)$; it is full with respect to ρ if it is left and right full. The reason for considering full maps with respect to a projective rank function is that the only maps to have a chance of becoming inverted under a homomorphism to a simple artinian ring are the full maps with respect to the induced projective rank function; of course, it is in general rather unlikely that they all do; however, we shall find that for a hereditary ring there are homomorphisms for each projective rank function that invert all full maps.

We see that entirely the same theory may be set up on the dual category of f.g. right projectives, where the rank of a f.g. right projective module is that of its dual module. It is clear that the dual of a left full map is right full and vice versa.

The important fact about the inner projective rank on a weakly semihereditary ring is an analogue of Sylvester's law of nullity. We say that a ring satisfies the law of nullity with respect to ρ, or, alternatively, that the projective rank function ρ is a Sylvester projective rank function, if for every pair of maps between f.g. projectives $\alpha : P_0 \to P_1$, $\beta : P_1 \to P_2$ such that $\alpha\beta = 0$, then $\rho(\alpha) + \rho(\beta) \leq \rho(P_1)$. If R is a ring such that all f.g. projectives are free of unique rank, and this rank is a Sylvester projective rank function, R is a Sylvester domain.

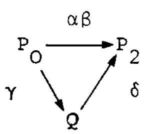
Theorem 1.11 Let R be a weakly semihereditary ring with a rank function ρ; then ρ is a Sylvester projective rank function.

Proof: Recall that if $\alpha\beta = 0$ for $\alpha : P_0 \to P_1$, and $\beta : P_1 \to P_2$ over a weakly semihereditary ring, then $P_1 \cong P' \oplus P''$, where the image of α lies in P', and the kernel of β contains P', so that β factors through P''. Hence $\rho(\alpha) \leq \rho(P')$, and $\rho(\beta) \leq \rho(P'')$, so that $\rho(\alpha) + \rho(\beta) \leq \rho(P_1)$.

There are a few results that we can deduce from the law of nullity for a projective rank function on a ring. On the whole, they are a little technical, but since we shall need them later, it seems better to bore the reader now than to break up the flow of later proofs. Their point is to demonstrate the analogy between these rings with Sylvester projective rank functions and simple artinian rings with the standard rank function.

Lemma 1.12 Let R be a ring with a Sylvester projective rank function ρ; then for any pair of maps $\alpha : P_0 \to P_1$, $\beta : P_1 \to P_2$, $\rho(\alpha\beta) \geq \rho(\alpha) + \rho(\beta) - \rho(P_1)$. In particular, this holds for weakly semihereditary rings for any projective rank function.

Proof: Suppose that



is a commutative diagram. Then we have the maps $(\alpha|\gamma) : P_0 \to P_1 \oplus Q$, $\begin{pmatrix}\beta \\ -\delta\end{pmatrix} : P_1 \oplus Q \to P_2$, and $(\alpha|\gamma)\begin{pmatrix}\beta \\ -\delta\end{pmatrix} = 0$;