KNOT TABULATIONS AND RELATED TOPICS

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0 Introduction

The aim of this article is to examine some of the ideas connected with the problem of classifying 1-dimensional knots. As the title suggests, the flavour is intended to be rather pragmatic. We follow through the history of knot tabulations, from the pre-topological dark ages of the last century up to the present day.

Dowker developed an interest in knot tabulations in the 1960's, and collaborated recently with the present author in the classification of knots of up to 13 crossings.

It is hoped that the expository parts of sections 1, 3 and 4 will make the article reasonably self-contained. We pay only cursory attention to "abelian" knot theory, as this is dealt with extensively in Cameron Gordon's survey article **(Gor)**.

I would like to thank John Conway, Raymond Lickorish and Larry Siebenmann for valuable conversations.

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Excerpt
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1 Preliminaries

A *knot* is a submanifold of S^3 homeomorphic to S^1. Below are drawings of three famous knots.

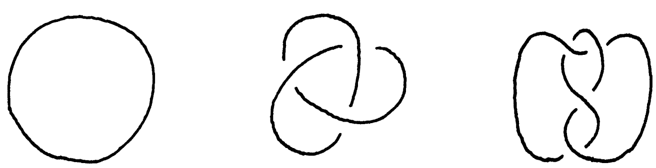

Fig. 1.1. The unknot The trefoil The figure-eight

Two knots K , L are *equivalent* if there exists an autohomeomorphism of S^3 mapping K onto L . The equivalence class of K is its *knot type*. We shall consider exclusively knot types with piecewise linear (or, equivalently, smooth) representatives, in order to steer clear of the virtually unexplored world of *wild knots*, i.e. knot types with no piecewise linear representative. The reader interested in wild knots can find information and references in **(Fox$_3$)**. We shall often use the word "knot" as an abbreviation for "knot type", when there is no danger of confusion. Thus the term "trefoil knot" really refers to the knot type of the knot depicted above, rather than the specific knot.

Choosing orientations of S^3 or K leads to stronger notions of knot equivalence. Let us suppose that an orientation of S^3 has been chosen. Then two knots K , L are equivalent via an orientation-preserving homeomorphism of S^3 if and only if they are *isotopic*, in the sense that there exists a homotopy $h_t : S^3 \to S^3$ $(0 \leqslant t \leqslant 1)$ such that $h_0 = 1$, each h_t is a homeomorphism, and $h_1(K) = L$.

This leads us to consideration of symmetries of knots. Let $\overline{K}$ be a mirror-image of K . If it happens that K can be isotoped to $\overline{K}$, then the above two notions of equivalence coincide for K , and K is said to be *amphicheiral*. The reader can check that the figure-eight knot is amphicheiral; the trefoil is not, though this of course requires proof. If, in addition, we orient the actual knots, an even stronger version of equivalence results: we can insist that the homeomorphism of S^3 mapping K to L preserves orientations of the knots, as well as that of S^3 . In this way, we also get finer notions of symmetry. Let K' be K with the orientation of the knot reversed. If K can be isotoped to K' , the knot K is said to be *invertible*. Both the trefoil and the figure-eight are readily seen to be invertible, by marking arrowheads on the knots and turning them over. For oriented knots, there are really two kinds of amphicheirality, depending on whether K can be isotoped to $\overline{K}$ or to $\overline{K}'$; Fox calls these + and - amphicheirality **(Fox$_3$)**. In practice, it is easy to find topological invariants which will detect non-amphicheirality, and it is exceedingly hard to detect non-invertibility. Significantly, Dehn **(De)** proved in 1914 that the trefoil was not amphicheiral, but the first proof of the existence of *any* non-invertible knots was due to Trotter **(Tro)** in 1964. For the important class of algebraic knots, these symmetry problems have been settled by Bonahon and Siebenmann.

Much analysis of knots is conveniently carried out in a more or less

2-dimensional setting. Let us regard S^3 as $\mathbb{R}^3 + \infty$. Then any knot is isotopic under an arbitrarily small deformation to a knot K in $\mathbb{R}^3$, whose image under the parallel projection $(x,y,z) \to (x,y,0)$ is a closed curve in the plane with no multiple points other than finitely many transverse double points. K is said to be in *regular position* with respect to this projection, and the image of K in the plane is called a *regular projection* of K . When drawing such a closed curve, if we leave suitable small gaps as in Fig. 1.1, we can indicate at each double point which of the two pre-image points on K has the larger z-coordinate, thus recovering completely the knot type, and indeed the isotopy class, of K . We can think of each "gap" as an open arc in the curve, coloured differently (white in this instance!). Such an embellished regular projection is called a *knot diagram* of K . Although knot diagrams obscure much of the 3-dimensional "feel" of knots, they are of undoubted use in computing knot invariants, and in tabulating knots.

The reader might well prefer a crisper approach to knot diagrams. We can define a knot diagram in the oriented plane to be a regular knot projection with signed crossing points. The intended interpretation of the signs is that a *positive* crossing conforms to the picture  with respect to some orientation of the closed curve, and a *negative* crossing conforms to the picture  . Note that the choice of orientation of the curve is immaterial here.

Any knot diagram drawn in the plane $z = 0$ can of course be converted into a corresponding knot in $\mathbb{R}^3$ by substituting, for each pair of gaps adjacent to a crossing, a semicircle pointing downwards:-

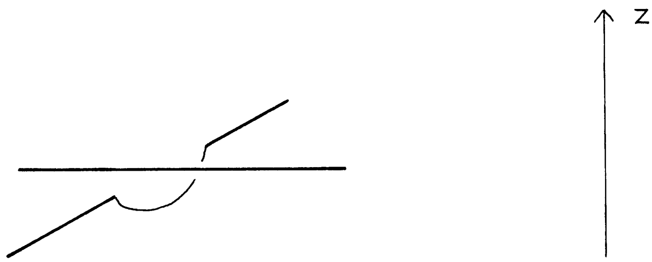

Fig. 1.2.

An important observation is that a diagram of a knot K can be deformed by an isotopy in the *extended* plane $\mathbb{R}^2 + \infty$ without losing the isotopy class of K . To see why this is so, observe that a diagram drawn on the 2-sphere of radius 1, centre (0,0,1) can be converted into a knot by replacing each adjacent pair of gaps by an outward-pointing semicircle, and that this realization is compatible with the realization for diagrams in the plane, via stereographic projection. For example, the following three diagrams represent isotopic knots.

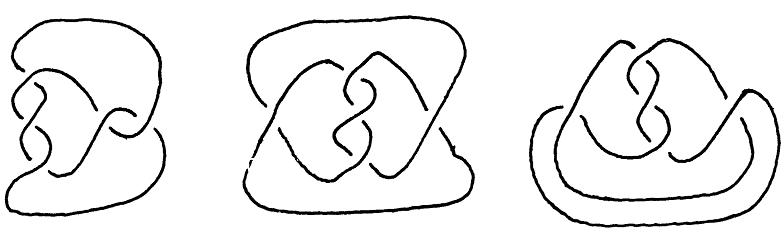

Fig. 1.3.

It now seems sensible to say that two diagrams in the plane are *equivalent* if there is a homeomorphism of the extended plane mapping one onto the other, preserving signs of crossings, and that they are *isotopic* if they are equivalent via an orientation-preserving homeomorphism of the extended plane. Equivalent (respectively isotopic) diagrams represent equivalent (respectively isotopic) knots.

The converse of this last statement is of course far from true. For instance, the crossings of any regular knot projection may be given signs so as to make a diagram of the unknot! Also, illustrated in Fig. 2.6 is a knot with no diagram of fewer than 13 crossings, yet with 769 inequivalent 13-crossing diagrams. However, if diagrams D_1, D_2 represent isotopic knots, D_1 can be transformed to D_2 by means of a finite sequence of so-called *Reidemeister moves*:-

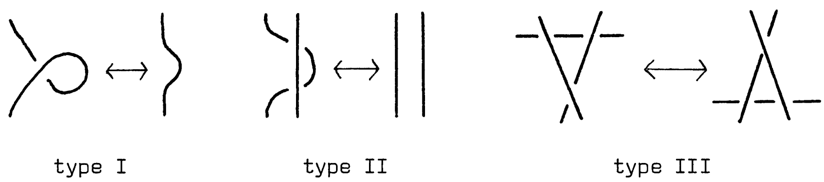

Fig. 1.4.

Proof of this fact is elementary, if somewhat tedious, and relies on standard techniques in PL topology for adjusting maps so as to simplify sets of self-intersection. For this reason, a proof is not included here. Of course, the Reidemeister moves do not give us an effective procedure for deciding whether two diagrams represent equivalent knots.

We now turn our attention to the factorization of knots. A knot K in S^3 is *prime* if any 2-sphere which meets K transversely in two points bounds one and only one 3-ball intersecting K in an unknotted spanning arc. That is, the ball-arc pair in question is homeomorphic to the standard pair (B,A), where $B = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| \leqslant 1\}$ and $A = \{(t,0,0) \in \mathbb{R}^3 : -1 \leqslant t \leqslant 1\}$. Thus, the unknot is not prime. Many knots are known to be prime, but proving that a given knot is prime can be a task which is not altogether trivial. A non-trivial knot which is not prime is *composite*; in layman's terms, a composite knot results from tying two knots separately in the same piece of string, but the validity of this interpretation depends on the theorem that two knots combined in this way cannot "cancel each other out", to produce the unknot. One of the many existing proofs of this "non-cancellation theorem" is given in §3.

In order to define the *connected sum*, or *composite* of two knots unambiguously, it is necessary to work in the oriented category. Let K , L be oriented knots in oriented S^3 . Let S^2_K , S^2_L bound 3-balls B^3_K , B^3_L which intersect K , L respectively in unknotted spanning arcs. Let S^2_K , S^2_L meet K , L respectively in 0-spheres denoted S^0_K , S^0_L . All these spheres inherit orientations from those of S^3 , K , L . Then the 3-sphere obtained by attaching $S^3 - \mathring{B}^3_K$ to $S^3 - \mathring{B}^3_L$ by an orientation-reversing homeomorphism $h: (S^2_L , S^0_L) \to (S^2_K , S^0_K)$ contains a knot $K \# L$, formed by joining $K \cap (S^3 - \mathring{B}^3_K)$ to $L \cap (S^3 - \mathring{B}^3_L)$. $K \# L$ is the *connected sum* of K and L , and is unique up to oriented equivalence. The reason for making h orientation-reversing is that $(S^3 , K \# L)$ then inherits an orientation naturally from the oriented pairs (S^3 , K) and (S^3 , L) .

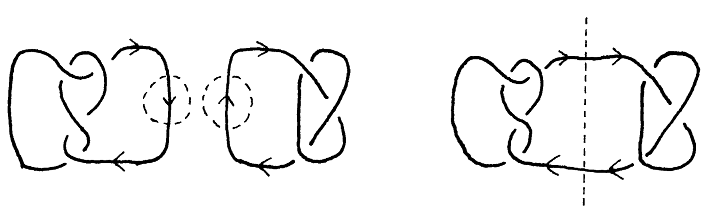

Fig. 1.5. K L K#L

Schubert proved in 1949 **(Schu$_1$)** that factorization of knots into primes is unique with respect to $\#$, up to order of the factors. Since the trefoil T is not amphicheiral, it follows that the knots $T \# T$ (the *granny* knot), $T \# \overline{T}$ (the *reef* or *square* knot), and $\overline{T} \# \overline{T}$ are mutually inequivalent in the oriented category (there is no need to specify a particular orientation of T itself as the trefoil is invertible). Since $\overline{T \# T} = \overline{T} \# \overline{T}$, it follows that the granny knot and the reef knot are not equivalent even in the weak (unoriented) sense. This last fact was proved by Seifert in 1933 **(Sei$_1$)**. Using a similar argument, if K , L are neither amphicheiral nor invertible, the knots $K \# L$, $K \# \overline{L}$, $K \# L'$, $K \# \overline{L}'$ are pairwise inequivalent in the weak sense.

The formation of connected sums of knots is subsumed in a different, more general construction, namely that of the satellites of a knot K . Working, as usual, in the PL-category, let h be an embedding of the solid torus $V^3 = S^1 \times D^2$ in S^3 such that the image of the *core* $S^1 \times \{0\}$ of V^3 is K . Let C be any simple closed curve in V^3 , which is not contained in any 3-ball in V^3 . Then h(C) is called a *satellite* of K , and K is a *companion* of h(C) . The study of companionship was initiated by Schubert in 1953 **(Schu$_2$)**, and recently Jaco, Shalen and Johannson **(JS, Joh)**, using some of Schubert's ideas, proved the important *characteristic variety theorem*, one of whose many consequences is that, essentially, knots factorize uniquely into companions. For further details, see **(BS)**. Illustrated in Fig. 1.6 are three satellites of the trefoil: a connected sum, a double and a cable (see **(Rol)** for details).

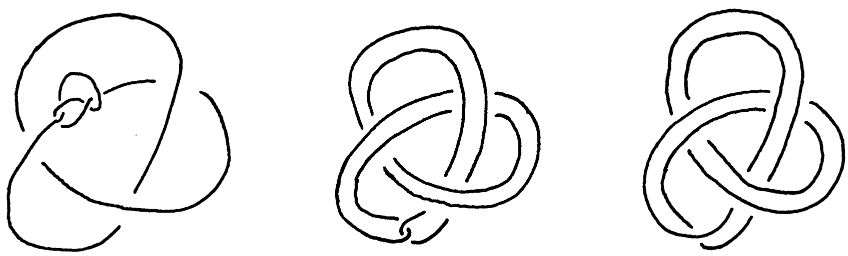

Fig. 1.6.

An important class of knots is that of alternating knots. Let us say that a diagram is *alternating* if (i) it has at least one crossing; (ii) the curve goes alternately over and under at successive crossing points; (iii) the diagram has no "nugatory" crossing  , where the shaded disc represents a portion of knot diagram which may or may not be trivial. Then an *alternating knot* is a knot admitting an alternating diagram. It is not difficult to see that every singular closed curve in the plane, which is a regular projection with at least one crossing of a knot, and in which there is no occurrence of  , is a regular projection of some alternating knot. Also, it is clear that such a curve must in fact have at least three crossing points. However, it is by no means obvious that there exist any knots which are not alternating. Bankwitz proved in 1930 that the *determinant* of a knot (see §4) is not exceeded by the number of crossings of any alternating diagram of the knot. It follows that the unknot, having determinant 1, is not alternating. Also, the 8-crossing torus knot, listed as 8_{19} in **(AB)** and denoted $T_{3,4}$ in §5 of this article, is a non-trivial non-alternating knot for the following reasons: its determinant is 3, there does not exist a non-trivial knot apart from the trefoil with a diagram of up to 3 crossings (the reader can quickly verify this), and $T_{3,4}$ is distinguished from the trefoil by its Alexander polynomial (again, see §5).

One of the most commonly used measures of complexity of a knot is its *crossing-number* $c(K)$. If D is a diagram of K, let $c(D)$ be the number of crossing points of D. Then $c(K) = \inf\{c(D) : D \text{ is a diagram of } K\}$. The advantage of this measure is that there are only finitely many diagrams, hence only finitely many knots of a given crossing-number. Standard practice when tabulating knots is to list all prime knots of up to a certain crossing-number. That tabulating knots is a precarious undertaking is evidenced by the numerous errors in published tables. For instance, a duplication in