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J. WLOKA

University of Kiel

TRANSLATED BY C.B AND M.J. THOMAS



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Contents

<i>Preface</i>	ix
I Sobolev spaces	1
§1 Notation, basic properties, distributions	1
1.1 <i>Notation</i>	1
1.2 <i>Partition of unity</i>	4
1.3 <i>Regularisation of functions</i>	8
1.4 <i>Distributions</i>	10
1.5 <i>The support of a distribution</i>	12
1.6 <i>Differentiation and multiplication</i>	14
1.7 <i>Distributions with compact support</i>	18
1.8 <i>Convolution</i>	20
1.9 <i>The Fourier transformation</i>	25
§2 Geometric assumptions for the domain Ω	35
2.1 <i>Segment and cone properties</i>	36
2.2 <i>The $N^{k,\kappa}$-property of Ω</i>	38
2.3 <i>(k, κ)-diffeomorphisms and (k, κ)-smooth Ωs</i>	47
2.4 <i>Normal transformations</i>	52
2.5 <i>Differentiable manifolds</i>	58
§3 Definitions and density properties for the Sobolev–Slobodeckii spaces $W^l_2(\Omega)$	61
3.1 <i>Definition of the Sobolev–Slobodeckii spaces $W^l_2(\Omega)$</i>	61
3.2 <i>Density properties</i>	64
§4 The transformation theorem and Sobolev spaces on differentiable manifolds	74
4.1 <i>The transformation theorem</i>	75
4.2 <i>Sobolev spaces on differentiable manifolds, and on the frontier $\partial\Omega$ of a (k, κ)-smooth region</i>	87
§5 Definition of Sobolev spaces by the Fourier transformation and extension theorems	90
5.1 <i>Sobolev spaces and the Fourier transformation</i>	91
5.2 <i>Extension theorems</i>	95
§6 Continuous embeddings and Sobolev’s lemma	105

§7 Compact embeddings	112
§8 The trace operator	120
§9 Weak sequential compactness and approximation of derivatives by difference quotients	133
II Elliptic differential operators	139
§10 Linear differential operators	139
§11 The Lopatinskiĭ–Šapiro condition and examples	148
11.1 <i>The Lopatinskiĭ–Šapiro condition</i>	148
11.2 <i>Examples</i>	157
§12 Fredholm operators	165
12.1 <i>The Riesz–Schauder spectral theorem (compact operators)</i>	165
12.2 <i>Fredholm operators</i>	168
12.3 <i>A priori estimates, the Weyl lemma and smoothable operators</i>	180
§13 The main theorem and some theorems on the index of elliptic boundary value problems	186
13.1 <i>The main theorems for elliptic boundary value problems</i>	186
13.2 <i>The index and spectrum of elliptic boundary value problems</i>	209
§14 Green’s formulae	213
14.1 <i>Normal boundary value operators and Dirichlet systems</i>	214
14.2 <i>The first Green formula</i>	219
14.3 <i>Adjoint boundary value operators and boundary value spaces</i>	222
14.4 <i>The second Green formula</i>	231
14.5 <i>The antidual operator L' and the adjoint boundary value problem</i>	235
§15 The adjoint boundary value problem and the connection with the image space of the original operator	239
§16 Examples	252
III Strongly elliptic differential operators and the method of variations	261
§17 Gelfand triples, the Lax–Milgram theorem, V-elliptic and V-coercive operators	261
17.1 <i>Gelfand triples</i>	261
17.2 <i>Representations for functionals on Sobolev spaces</i>	268
17.3 <i>The Lax–Milgram theorem</i>	271
17.4 <i>V-elliptic and V-coercive forms, solution theorems</i>	273
17.5 <i>The Green operator</i>	275
17.6 <i>The concepts V-elliptic and V-coercive for differential operators</i>	279
§18 Agmon’s condition	280
§19 Agmon’s theorem: conditions for the V-coercion of strongly elliptic differential operators	290
19.1 <i>The theorems of Gårding and Agmon</i>	290
19.2 <i>Examples, including the Dirichlet problem for strongly elliptic differential operators</i>	302

§20	Regularity of the solutions of strongly elliptic equations	307
§21	The solution theorem for strongly elliptic equations and examples	336
§22	The Schauder fixed point theorem and a non-linear problem	361
§23	Elliptic boundary value problems for unbounded regions	370
IV	Parabolic differential operators	376
§24	The Bochner integral	376
24.1	<i>Pettis' theorem</i>	376
24.2	<i>The Bochner integral</i>	384
§25	Distributions with values in a Hilbert space H and the space $W(0, T)$	390
§26	The existence and uniqueness of the solution of a parabolic differential equation	395
§27	The regularity of solutions of the parabolic differential equation	403
27.1	<i>An abstract regularity theorem</i>	404
27.2	<i>Differentiability with respect to t</i>	411
27.3	<i>Differentiability with respect to x, respectively t</i>	414
§28	Examples	423
V	Hyperbolic differential operators	434
§29	Existence and uniqueness of the solution	434
§30	Regularity of the solutions of the hyperbolic differential equation	442
30.1	<i>An abstract regularity theorem</i>	442
30.2	<i>Differentiability with respect to t</i>	445
30.3	<i>Differentiability with respect to x</i>	447
§31	Examples	452
VI	Difference processes for the calculation of the solution of the partial differential equation	462
§32	Functional analytic concepts for difference processes	462
§33	Difference processes for elliptic differential equations and for the wave equation	481
33.1	<i>Some important inequalities</i>	481
33.2	<i>Construction of a difference process for the Dirichlet problem</i>	484
33.3	<i>A difference process for the wave equation in several space variables</i>	488
§34	Evolution equations	496
34.1	<i>The time-independent case</i>	498
34.2	<i>The time-dependent case</i>	503
34.3	<i>Stability behaviour of the perturbed process</i>	505
34.4	<i>Several step processes</i>	507
	References	511
	Function and distribution spaces	515
	Index	516

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Frontmatter
[More information](#)

For Brigitte

Preface

Boundary value problems are the subject of this book. All boundary value conditions for elliptic differential operators are given, using the Lopatinskiĭ–Šapiro condition (= covering condition), which lead to the normal solvability of a boundary value problem. The variational method is also presented in detail, and questions of its connections with general elliptic theory considered. Those parabolic and hyperbolic equations for which the right-hand side (derivatives with respect to x) is an elliptic differential operator are considered, and the knowledge about elliptic operators is used in order to obtain insight into the solvability and regularity properties of the solution for mixed problems.

I have chosen a form of the Lopatinskiĭ–Šapiro condition which allows us to test immediately, whether or not given boundary value conditions satisfy it. It appears that all classical boundary value problems satisfy it, the examples are worked through individually.

In order not to overexpand the compass of the book, and to maintain its introductory character, I have not considered pseudo-differential operators; all the same I have proved the main theorem for elliptic boundary value problems by means of pseudo-differential operators – without calling them such.

Before the discussion of differential equations there is an introductory chapter on distributions and Sobolev spaces; here I have proceeded in an elementary way, working with the Fourier transformation, and not using interpolation theorems. This is possible without further assumptions as long as one remains inside L^2 -theory. I have not considered the L^p -theory; this comes into its own for non-linear equations, see for example Lions [3], while in the linear case it does not bring any essentially new insights.

I have looked very precisely into the differentiability properties of the

frontier $\partial\Omega$; in the end we want to solve the Dirichlet problem on the square as well as on the circle, and the proofs are not essentially simpler for the C^∞ -theory.

Among the many procedures for the practical solution of a partial differential equation I have singled out the difference process – its appeal lies in its simplicity: derivatives are replaced by difference quotients and so we obtain a system of linear equations, which in the main we can solve by the usual methods. In the last chapter I wish less to lead into modern methods, than to transmit the feeling to the reader that it is actually possible numerically to solve partial differential equations.

The reader ought to be familiar with the language of functional analysis, at about the level of the books of Heuser [1], or Wloka [1]. He may find the basic theorems of functional analysis relevant for analysis, such as the Hahn–Banach, Banach–Steinhaus and Riesz theorems, the open mapping theorem . . . , collected and proved on pp. 12–27 in L.H. Loomis, *An introduction to abstract harmonic analysis*, New York (1953). In separate sections I have thoroughly considered less familiar material such as, for example, the theory of Fredholm operators, Gelfand triples, abstract Green solution operators, the Schauder fixed point theorem and the Bochner integral. In this way I hope to spare the reader a time-consuming hunt through the literature. Another aim, which I have included in these functional analytic sections, is wherever possible to replace hard analysis by soft analysis, and so restricted the difficult estimation machinery to an unavoidable minimum. In this way I also believe that I have given the reader a better view of the connections and possible generalisations.

I owe a very heavy debt of gratitude to R. Mennicken, G. Bauer and B. Sagraloff from Regensburg, to my students J. Benner, R. Janssen, R. Rath, and to Miss Inga Haecks. They have read the complete manuscript critically and carefully, and have provided many important remarks and improvements. I am particularly grateful to M. König for assistance in the laborious proof reading of the German edition.

I thank Professor Dr G. Köthe for encouraging me to write this book, and the publishers both for their assistance in the preparation of the manuscript and cooperation in its presentation.

Kiel 1980

J. Wloka

Note added in the English translation

I have made some corrections and slight changes, especially in §§ 14 and 15. To English-speaking readers, instead of the books of Heuser [1] and Wloka [1] (see above) I would recommend those of Schechter [1] or Taylor [1] as an introduction to the language of functional analysis.

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J. Wloka