I

Sobolev spaces

§1 Notation, basic properties, distributions

In order to build up the theory of Sobolev spaces in a simple manner we need a generalised notion of differentiation; this we find in L. Schwartz' theory of distributions, and give here a short introduction to the theory of distributions, in so far as it is applicable to Sobolev spaces. In order to spare the reader additional labour we have been careful to give all proofs. Our development is concentrated on the following points: partition of unity, an aid which we shall use at many points in this book; the generalised notion of differentiation, for which we also present theorems illuminating the connection with classical differentiation; the regularisation (convolution) of functions and distributions, which play a great role in approximation and density problems; Fourier transformation, which we study in the spaces $\mathscr{S}$, $\mathscr{S}'$ and $L_2(\mathbb{R}^r)$, obtaining an important analytic tool, that will render us good service in many questions.

For similar introductions we refer to the books of Hörmander [1] and Rudin [3]; for a broad initiation into distribution theory the original book of L. Schwartz [1] is still to be recommended.

1.1 Notation

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{C}$ the set of complex numbers; we denote by $\mathbb{R}^r$ and $\mathbb{C}^r$ the real and complex r-dimensional spaces equipped with the Euclidean norm

$$|x| = [|x_1|^2 + \cdots + |x_r|^2]^{1/2}, \quad x = (x_1, \ldots, x_r) \in \mathbb{R}^r \text{ (respectively } \mathbb{C}^r).$$

We use L. Schwartz' notation for derivatives, products, faculties, etc. Let

$s = (s_1, \ldots, s_r)$ be a multi-index, $s_i \in \mathbb{N} = \{0, 1, \ldots\}$, $i = 1, 2, \ldots, r$, we write

$$\mathrm{D}^s = \frac{\partial^{s_1 + \cdots + s_r}}{\partial x_1^{s_1} \cdots \partial x_r^{s_r}}, \quad |s| = s_1 + \cdots + s_r, \quad \Delta = \sum_{j=1}^{r} \frac{\partial^2}{\partial x_j^2},$$

$$x^s = x_1^{s_1} x_2^{s_2} \cdots x_r^{s_r}, \quad 0^0 = 0,$$

$$s! = s_1! \cdots s_r!, \quad \binom{m}{s} = \binom{m_1}{s_1} \cdots \binom{m_r}{s_r}.$$

Let Ω be an open set in $\mathbb{R}^r$. The elements of $C^l(\Omega)$ are all (complex valued) functions $\varphi(x)$, $x \in \Omega$, which on Ω possess continuous and bounded derivatives $\mathrm{D}^s\varphi(x)$, $|s| \leqslant l$ (up to order l). We define the norm $\|\varphi\|_{C^l}$ on C^l by

$$\|\varphi\|_{C^l} = \sup_{\substack{|s| \leqslant l \\ x \in \Omega}} |\mathrm{D}^s\varphi(x)|.$$

Convergence in the space $C^l(\Omega)$ means uniform convergence in Ω not only of the sequence of functions itself, but also of the sequence of its sth partial derivatives ($|s| \leqslant l$). By $C^l(\bar{\Omega})$ we understand the proper subspace of $C^l(\Omega)$, consisting of all functions $\varphi \in C^l(\Omega)$ which, together with their derivatives up to order l, are also continuous on the frontier of Ω (that is, continuous on $\bar{\Omega}$). In the case that Ω is in addition bounded, we can use the norm

$$\|\varphi\|_{C^l(\bar{\Omega})} = \max_{\substack{|s| \leqslant l \\ x \in \bar{\Omega}}} |\mathrm{D}^s\varphi(x)|.$$

We denote by $\mathscr{E}^l(\Omega)$ the set of all functions which together with their derivatives up to order l are continuous on Ω. (Here no boundedness is asked for.) For $\mathscr{E}^\infty(\Omega)$ we also write $\mathscr{E}(\Omega)$ or $C^\infty(\Omega)$ – later we shall introduce on $\mathscr{E}(\Omega)$ the topology of an F-space.

We say that a function φ is λ-Hölder continuous on Ω; if

$$\frac{|\varphi(x) - \varphi(y)|}{|x - y|^\lambda} \leqslant C < \infty$$

holds for all $x, y \in \Omega$; here $0 < \lambda \leqslant 1$ and $x \neq y$. (The λ-Hölder continuous functions on Ω for $\lambda > 1$ are constant.) In the case $\lambda = 1$ we also talk of Lipschitz continuous functions. We define the space $C^{l,\lambda}(\Omega)$ to be the collection of all l-fold continuous, differentiable, bounded functions φ on Ω (also let all derivatives $\mathrm{D}^s\varphi$, $|s| \leqslant l$ be bounded on Ω) for which the lth derivatives are λ-Hölder continuous. As norm in $C^{l,\lambda}(\Omega)$ we take the expression

$$\|\varphi\|_{l,\lambda} = \sup_{\substack{|s| \leqslant l \\ x \in \Omega}} |\mathrm{D}^s\varphi(x)| + \sup_{\substack{|s| = l \\ y, x \in \Omega \\ x \neq y}} \frac{|\mathrm{D}^s\varphi(x) - \mathrm{D}^s\varphi(y)|}{|x - y|^\lambda}.$$

All the spaces introduced are complete, and hence are Banach or Fréchet spaces; the simple proofs are left to the reader, see also Wloka [1], pp. 21 and 55. For the sake of completeness we define

$$C^{0,0}(\Omega):=C(\Omega), \quad C^{l,0}(\Omega):=C^l(\Omega).$$

We wish to define the $L_p(\Omega)$ spaces. First suppose that $1 \leqslant p < \infty$. The space $L_p(\Omega)$ consists of all Lebesgue measurable functions defined on $\Omega \subset \mathbb{R}^r$, whose pth power is integrable with respect to the Lebesgue measure $dx = dx_1 \cdots dx_r = d\mu$, that is,

$$\int_\Omega |\varphi(x)|^p \, dx < \infty \text{ holds.}$$

As norm in $L_p(\Omega)$ we take the expression

$$\|\varphi\|_p = \left[\int_\Omega |\varphi(x)|^p \, dx\right]^{1/p}.$$

Strictly speaking, in $L_p(\Omega)$ we have to deal not with single functions but with classes of functions, which differ from each other only on sets of measure zero. Thus if $\|\varphi\|_p = 0$, it follows that $\varphi(x) = 0$ almost everywhere on Ω, but not that $\varphi(x)$ is identically zero. We come now to the definition of the space $L_\infty(\Omega)$, $p = \infty$. This space consists of all functions φ defined on Ω which are Lebesgue measurable and bounded almost everywhere (φ is called bounded almost everywhere if the inequality $|\varphi(x)| \leqslant M < \infty$ holds outside a set $\{x : x \in A\}$ of measure zero). By means of

$$\|\varphi\|_\infty = \operatorname*{supess}_{x\in\Omega} |\varphi(x)| =: \inf_{\substack{A\subset\Omega \\ \mu(A)=0}} \sup_{x\in\Omega\setminus A} |\varphi(x)|,$$

we introduce a norm on $L_\infty(\Omega)$. We can show that the spaces $L_p(\Omega)$, $1 \leqslant p \leqslant \infty$ are complete, see for example Wloka [1], p. 52.

The set $L_1^{\text{loc}}(\Omega)$ consists of all Lebesgue measurable functions on Ω, which are integrable on each compact subset $K \subset \Omega$, that is,

$$\int_K |\varphi(x)| \, dx < \infty \quad \text{for all } K \subset\subset \Omega.$$

It is easy to show that all the C-spaces and L_p-spaces introduced here are contained in $L_1^{\text{loc}}(\Omega)$.

The space $L_2(\Omega)$ is particularly important in what follows; it is a separable Hilbert space with the inner product

$$(\varphi, \psi) = \int_\Omega \varphi(x) \cdot \overline{\psi(x)} \, dx; \quad \varphi; \psi \in L_2(\Omega).$$

For embedding theory we need a compactness criterion for L_p-spaces, this goes back to Kolmogorov; for the proof, see for example Wloka [1], p. 201.

Kolmogorov compactness criterion. Let M be a subset of the space $L_p(\Omega)$, $1 \leqslant p < \infty$. M is relatively compact if and only if the following three conditions are satisfied:

1. M is bounded in $L_p(\Omega)$, that is, $\sup_{\varphi \in M} \|\varphi\|_p = \sup_{\varphi \in M} [\int_\Omega |\varphi(x)|^p \mathrm{d}x]^{1/p} < \infty$.
2. $\lim_{h \to 0} \int_\Omega |\varphi(x+h) - \varphi(x)|^p \mathrm{d}x = 0$ holds uniformly for $\varphi \in M$.
3. $\lim_{\alpha \uparrow \beta} \int_{\{|x| > \alpha\} \cap \Omega} |\varphi(x)|^p = 0$ holds uniformly for $\varphi \in M$.

If the domain Ω is bounded, then condition 3 is superfluous.

1.2 Partition of unity

The partition of unity is known to be an important tool; we shall often make use of it in what follows. In order to introduce it we need some definitions. We say that $\{\Omega_i, i \in I\}$ – the indexing set I is arbitrary – is an open cover of $\mathbb{R}^r$, if the sets Ω_i are open and

$$\bigcup_{i \in I} \Omega_i = \mathbb{R}^r \text{ holds.} \tag{1}$$

We call a cover $\{U_j, j \in J\}$, a refinement of $\{\Omega_i, i \in I\}$, provided that for each j we can find some $i(j)$ with

$$U_j \subset \Omega_{i(j)}. \tag{2}$$

We say that a cover $\{\Omega_i, i \in I\}$ is locally finite, if for each point $x \in \mathbb{R}^r$ we can find a ball $B(x, \rho)$ for which $\Omega_i \cap B(x, \rho) \neq \varnothing$ at most finitely often. We want to show that $\mathbb{R}^r$ is paracompact, which means that for each open cover $\{\Omega_i, i \in I\}$ we can find a locally finite refinement $\{U_j, j \in J\}$. However, we prove much more:

Theorem 1.1 *Let $\{\Omega_i, i \in I\}$ be an open cover of $\mathbb{R}^r$, then there exists a locally finite open refinement $\{U_j, j \in \mathbb{N}\}$ with countable indexing set $J = \mathbb{N}$, and with $\bar{U}_j$ compact, $j \in \mathbb{N}$.*

Proof. Let B_n be the ball $B(0, n)$, $n = 1, 2, \ldots$; we cover $\mathbb{R}^r$ by the compact annuli

$$\bar{B}_1, \bar{B}_2 \backslash B_1, \ldots, \bar{B}_{n+1} \backslash B_n, \ldots.$$

Let $x \in \bar{B}_1$. Because of (1), x lies in some Ω_{i_x}, and we may choose $U_x = B(x, \rho)$ in such a way that

$$\bar{U}_x \subset \Omega_{i_x} \quad \text{and} \quad \rho < \tfrac{1}{2}. \tag{3}$$

The U_x form an open cover of the compact set $\bar{B}_1$, so already finitely many $U_1, \ldots, U_m$ cover the ball $\bar{B}_1$. In this way we have defined the refinement $U = \{U_j\}$ for $j = 1, 2, \ldots, m$. We next consider the annulus $\bar{B}_2 \backslash B_1$ and proceed.

We choose V_x with

$$\bar{V}_x \subset \Omega_{i_x}, \quad V_x = B(x, \rho), \rho < \tfrac{1}{2}, x \in \bar{B}_2 \backslash B_1. \tag{3'}$$

Since $\bar{B}_2 \backslash B_1$ is compact we can select a finite subcover $\{V_1, \ldots, V_l\}$ from the cover $\{V_x\}$. We discard those Vs which already appear among $U_1, \ldots, U_m$ and call the remainder $U_{m+1}, \ldots, U_k$. $\{U_1, \ldots, U_k\}$ covers $\bar{B}_2$. Continuing in this way we find the open, countable cover $\{U_j, j \in \mathbb{N}\}$ of $\mathbb{R}^r$. Because of (3) $\{U_j\}$ is a refinement of $\{\Omega_i\}$, and because $\rho < \frac{1}{2}$ the sets $\bar{U}_j$ are compact. We show that $\{U_j\}$ is locally finite. Let $x \in \bar{B}_n \backslash B_{n-1}$ and consider the ball $B(x, \frac{1}{4})$. Because $\rho < \frac{1}{2}$ only the sets U_j which cover $\bar{B}_n \backslash B_{n-1}$ and the neighbouring annuli $\bar{B}_{n+1} \backslash B_n$ and $\bar{B}_{n-1} \backslash B_{n-2}$ can actually meet the ball $B(x, \frac{1}{4})$; and there are – by construction – only finitely many such U_js. ■

Remark 1.1 Theorem 1.1 also holds for each locally compact manifold, which satisfies the second axiom of countability (that is, $M = \bigcup_{n=1}^{\infty} K_n, K_n$ compact).

For the sake of completeness we present here the simple proof of Warner [1], p. 9.

Proof. We take the K_n in the second axiom of countability to be compact, $K_n \subset K_{n+1}$, $\bigcup_{n=1}^{\infty} K_n = M$, and prove first the existence of a sequence of open sets G_n with $\bar{G}_n$ compact, $\bar{G}_n \subset G_{n+1}$, $\bigcup_{n=1}^{\infty} G_n = M$.

Let $x \in K_1$, because of local compactness we may choose a neighbourhood U_x with $\bar{U}_x$ compact and such that $K_1 \subset \bigcup_{x \in K_1} U_x$. Therefore since K_1 is compact

$$K_1 \subset \bigcup_{i=1}^{m} U_{x_i}.$$

We set $G_1 := \bigcup_{i=1}^{m} U_{x_i}$ and define the other $G_n, n \geqslant 2$, inductively. If G_{n-1} is already defined, we consider the compact set $K_n \cup \bar{G}_{n-1}$ and we find as before locally compact neighbourhoods $U_{x_i}^n$ with

$$K_n \cup \bar{G}_{n-1} \subset \bigcup_{i=1}^{m_n} U_{x_i}^n.$$

We set $G_n := \bigcup_{i=1}^{m_n} U_{x_i}^n$, and can then easily verify the desired properties of G_n.

Now for the assertions of Theorem 1.1. Let $\{\Omega_i, i \in I\}$ be an arbitrary open

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cover of M. For $n \geqslant 3$ the set $\bar{G}_n \backslash G_{n-1}$ is compact and contained in the open set $G_{n+1} \backslash \bar{G}_{n-2}$. We consider the open cover

$$\{\Omega_i \cap (G_{n+1} \backslash \bar{G}_{n-2}), i \in I\} \text{ of } \bar{G}_n \backslash G_{n-1}$$

and select from it the finite cover

$$\bar{G}_n \backslash G_{n-1} \subset \bigcup_{j=1}^{m_n} U_j^n; \quad U_j^n = \Omega_j \cap (G_{n+1} \backslash \bar{G}_{n-2}).$$

Similarly we select from the open cover $\{\Omega_i \cap G_3, i \in I\}$ of $\bar{G}_2$ the finite cover

$$\bar{G}_2 \subset \bigcup_{j=1}^{m} U_j, \quad U_j = \Omega_j \cap G_3.$$

It is easily seen that the U's have the desired properties in Theorem 1.1, thus, for example, local finiteness: let $x \in M$, then there exists some $n \geqslant 3$ with $x \in G_n \backslash G_{n-1}$ (for $n = 2$ and $x \in G_2$), hence also $x \in \bar{G}_n \backslash G_{n-1}$. Since

$$x \in \bar{G}_n \backslash G_{n-1} \subset G_{n+1} \backslash \bar{G}_{n-2}$$

holds, and $G_{n+1} \backslash \bar{G}_{n-2}$ is open, we can find some neighbourhood U_x with

$$x \in U_x \subset G_{n+1} \backslash \bar{G}_{n-2}.$$

This U_x can only meet those U's that either lie in $G_{n+1} \backslash \bar{G}_{n-2}$ or in its neighbours, that is, in $G_{n+2} \backslash \bar{G}_{n-1}$, respectively $G_n \backslash \bar{G}_{n-3}$ (set $G_0 = \varnothing$). These are, however, by construction only finitely many in number, so that we have proved local finiteness (the proof for $n = 2$ is similar). ■

Corollary 1.1 *Let $\Omega \subset \mathbb{R}^r$ be open and let $\{\Omega_i, i \in I\}$ be an open cover of Ω. Then the assertions of Theorem* 1.1 *hold.*

It suffices to adjoin an additional set, for example $\Omega_0 = \mathbb{R}^r$, to the cover $\{\Omega_i\}$ in order to obtain a cover of $\mathbb{R}^r$, and be able to apply Theorem 1.1. We may also take balls $B(x_j, a_j)$ for the refinement $\{U_j\}$.

A continuous function $\varphi(x)$, $x \in \mathbb{R}^r$, defined on $\mathbb{R}^r$ is called finite if it vanishes outside a bounded subset. The support of φ, supp φ, is the closure of all points x with $\varphi(x) \neq 0$, in symbols

$$\operatorname{supp} \varphi := \overline{\{x : \varphi(x) \neq 0\}}.$$

Following L. Schwartz we denote the set of all finite, infinitely differentiable functions with supports in Ω (Ω open) by the symbol $\mathscr{D}(\Omega)$ (or $C_0^\infty(\Omega)$), and call the elements of $\mathscr{D}(\Omega)$ fundamental functions. For $\mathscr{D}(\mathbb{R}^r)$ we simply write $\mathscr{D}$.

Example Let $g(t), t\in\mathbb{R}^1$ denote the function

$$g(t)=\begin{cases}0 & \text{if } t\leqslant 0.\\ e^{-1/t} & \text{if } t>0,\end{cases}$$

which belongs to C^∞ but not to $\mathscr{D}$. With the help of $g(t)$ we can easily construct a fundamental function $h\in\mathscr{D}$:

$$h(x)=g(1-|x|^2)=\begin{cases}0 & \text{if } |x|\geqslant 1,\\ \exp(-1/1-|x|^2) & \text{if } |x|<1,\end{cases}$$

where $|x|^2=x_1^2+\cdots+x_r^2$. We have

$$\operatorname{supp} h=\overline{B(0,1)}.$$

For $a>0$, $x_0\in\mathbb{R}^r$, we set

$$\varphi_{x_0,a}(x)=h\left(\frac{x-x_0}{a}\right)\in\mathscr{D}(\mathbb{R}^r), \tag{4}$$

so that we have

$$\operatorname{supp}\varphi_{x_0,a}=\overline{B(x_0,a)},\quad\text{and}\quad \varphi_{x_0,a}(x)>0\quad\text{for } x\in B(x_0,a). \tag{5}$$

By a *partition of unity* $\{\alpha_j, j\in J\}$ on Ω we understand a class of functions $\alpha_j\in\mathscr{D}(\Omega)$ with the following properties:

(a) the collection of supports $\{\operatorname{supp}\alpha_j : j\in J\}$ is locally finite,
(b) $0\leqslant\alpha_j(x)\leqslant 1$, and $\sum_{j\in J}\alpha_j(x)\equiv 1$ for $x\in\Omega$.

Because of the local finiteness of the supports, the series appearing in (b) has only finitely many non-zero members for each $x\in\Omega$. We say that the partition of unity $\{\alpha_j, j\in J\}$ is subordinate to the cover $\{\Omega_i, i\in I\}$, if for each $j\in J$ there exists an $i(j)\in I$ with

$$\operatorname{supp}\alpha_j\subset\Omega_{i(j)}.$$

Theorem 1.2 (Partition of unity) *Let $\{\Omega_i, i\in I\}$ be an open cover of the open set $\Omega\subset\mathbb{R}^r$. There exists a partition of unity $\{\alpha_j, j\in\mathbb{N}\}$ subordinate to the cover $\{\Omega_i, i\in I\}$ with countable indexing set $J=\mathbb{N}$.*

Proof. By Theorem 1.1, Corollary 1.1, there exists a countable, locally finite refinement $\{U_j, j\in\mathbb{N}\}$ with $U_j=B(x_j,a_j)$ and $\bar{U}_j\subset\Omega_{i(j)}$. We take the functions (4), $\varphi_{x_j,a_j}\in\mathscr{D}(\Omega)$, and by (5) have $\operatorname{supp}\varphi_{x_j,a_j}=\bar{U}_j$. In the series $\psi(x)=\sum_j\varphi_{x_j,a_j}(x)$, because of local finiteness, there are only finitely many members differing from zero on $B(x,\rho)$. Hence $\psi\in C^\infty(\Omega)$, and by the covering property together with (5), $\psi(x)>0$ for all $x\in\Omega$. We define $\alpha_j(x)=\varphi_{x_j,a_j}(x)/\psi(x)$ and see that (a) and (b) are satisfied. ■

Complement We can also arrange that $\sum_j \beta_j^2(x) = 1$, $\beta_j \in \mathscr{D}(\Omega)$. We simply put

$$\beta_j(x) = \frac{\varphi_{x_j,a_j}(x)}{[\sum_j(\varphi_{x_j,a_j}(x))^2]^{1/2}}.$$

Corollary 1.2 *Let A be closed in $\mathbb{R}^r$ and G open with $A \subset G$. There exists an infinitely differentiable function α with the properties:*

(a) $0 \leqslant \alpha(x) \leqslant 1$ *for all* $x \in \mathbb{R}^r$,
(b) $\alpha(x) = 1$ *for* $x \in A$,
(c) $\operatorname{supp} \alpha \subset G$, *and in particular* $\alpha(x) = 0$ *on the complement of* G, $x \in \mathbf{C}G$.

Proof. We consider the cover $\{G, \mathbf{C}A\}$ of $\mathbb{R}^r$. Let $\alpha_j, j \in \mathbb{N}$ be a subordinate partition of unity (Theorem 1.2), that is, the support of α_j lies either in G or in $\mathbf{C}A$. We define α to be the sum of all those α_j, whose support lies in G. In short

$$M := \{j \in \mathbb{N} : \operatorname{supp} \alpha_j \subset G\}, \quad \alpha(x) = \sum_{j \in M} \alpha_j(x),$$

(a) is then clear. (c) follows from

$$\operatorname{supp} \alpha = \overline{\bigcup_{j \in M} \operatorname{supp} \alpha_j} = \bigcup_{j \in M} \operatorname{supp} \alpha_j \subset G, \tag{6}$$

where the equation in the middle holds for locally finite families. Now for (b) we rearrange

$$1 = \sum_j \alpha_j(x) = \sum_{j \in M} \alpha_j(x) + \sum{}' \alpha_j(x) = \alpha(x) + \sum{}' \alpha_j(x), \tag{7}$$

where $\sum' \alpha_j(x)$ is the sum of those α_j whose supports lie in $\mathbf{C}A$, but not in G. If $x \in A$, then $\sum' \alpha_j(x) = 0$, so that given (7) we have proved (b). ■

1.3 Regularisation of functions

We return once more to the function $h \in \mathscr{D}(\mathbb{R}^r)$ – see (4). We have

$$\int_{\mathbb{R}^r} h(x)\,\mathrm{d}x = C > 0,$$

and we set $\tilde{h}(x) = (1/C)h(x)$, so that we obtain

$$\int_{\mathbb{R}^r} \tilde{h}(x)\,\mathrm{d}x = \int_{|x| \leqslant 1} \tilde{h}(x)\,\mathrm{d}x = 1.$$

The function $h_\varepsilon(x) = (1/\varepsilon^r)\tilde{h}(x/\varepsilon)$ also has the property

$$\int_{\mathbb{R}^r} h_\varepsilon(x)\,\mathrm{d}x = 1,$$

and is a fundamental function which vanishes for $|x| \geqslant \varepsilon$. The function $h_\varepsilon(x)$

plays an important part in so-called *regularisation*. We describe the *convolution integral*

$$\varphi_\varepsilon(x) = (\varphi * h_\varepsilon)(x) = \int_{\mathbf{R}^r} \varphi(y)\cdot h_\varepsilon(x-y)\,\mathrm{d}y = \int_{\mathbf{R}^r} h_\varepsilon(y)(x-y)\,\mathrm{d}y \tag{8}$$

as the regularisation of φ.

Theorem 1.3 *Let φ be integrable (that is, $\varphi \in L_1$) and let φ vanish outside a compact subset K of Ω. We have the following assertions:*

(a) *The support of $\varphi_\varepsilon = \varphi * h_\varepsilon$ is contained in $K_\varepsilon = \{x : d(x, K) \leqslant \varepsilon\}$, and is again compact.*
(b) *If $\varepsilon < d(K, \mathbf{C}\Omega)$, then $\varphi_\varepsilon \in \mathscr{D}(\Omega)$.*
(c) *If $\varphi \in L_p(\Omega)$, $1 \leqslant p < \infty$, then $\lim_{\varepsilon\to 0} \| \varphi_\varepsilon - \varphi \|_p = 0$.*
(d) *If φ is continuous, then $\varphi_\varepsilon \to \varphi$ uniformly as $\varepsilon \to 0$.*

Proof. Formula (8) written as $\varphi_\varepsilon(x) = \int_K \varphi(y) h_\varepsilon(x-y)\,\mathrm{d}y$ shows that, because φ is integrable, the Lebesgue theorems on the interchange of integral and limit are applicable, and we obtain

$$\varphi_\varepsilon(x) \in C^\infty. \tag{9}$$

Let $\varphi_\varepsilon(x) \neq 0$; by formula (8) it must be that $x - y \in K$ and $|y| \leqslant \varepsilon$. If we write $K_\varepsilon := \{x : d(x, K) \leqslant \varepsilon\}$, we deduce that x must belong to K_ε, that is the support of φ_ε is a closed subset of K_ε, and we have proved (a), (b) follows immediately from (a) and (9). Suppose now that $\varphi \in L_p(\Omega)$, $1 < p < \infty$; the Hölder inequality (Schwarz inequality in the case $p = 2$) applied to (8) gives $(1 = 1/p + 1/q)$

$$\begin{aligned} \|\varphi_\varepsilon\|_p^p &\leqslant \int_\Omega \left[\int_{\mathbf{R}^r} |\varphi(y)|\, h_\varepsilon^{1/p+1/q}(x-y)\,\mathrm{d}y \right]^p \mathrm{d}x \\ &\leqslant \int_\Omega \left[\int_{\mathbf{R}^r} |\varphi(y)|^p h_\varepsilon^{p/p}(x-y)\,\mathrm{d}y \right]^{p/p} \left[\int_{\mathbf{R}^r} h_\varepsilon^{q/q}(x-y)\,\mathrm{d}y \right]^{p/q} \mathrm{d}x \\ &= \int_{\mathbf{R}^r} |\varphi(y)|^p \left(\int_\Omega h_\varepsilon(x-y)\,\mathrm{d}x \right) \mathrm{d}y = \|\varphi\|_p^p \quad \text{for } \varepsilon < d(K, \mathbf{C}\Omega), \end{aligned} \tag{10}$$

that is, φ_ε belongs to $L_p(\Omega)$. The Hölder inequality applied in the same way as in (10) gives

$$\begin{aligned} \|\varphi_\varepsilon - \varphi\|_p^p &= \int_\Omega |h_\varepsilon * \varphi(x) - \varphi(x)|^p\,\mathrm{d}x \leqslant \int_\Omega \left[\int_{\mathbf{R}^r} h_\varepsilon(x-y)|\varphi(y) - \varphi(x)|\,\mathrm{d}y \right]^p \mathrm{d}x \\ &\leqslant \int_\Omega \int_{|y| \leqslant \varepsilon} h_\varepsilon(y) |\varphi(x-y) - \varphi(x)|^p\,\mathrm{d}y\,\mathrm{d}x \\ &\leqslant \sup_{y \leqslant \varepsilon} \int_\Omega |\varphi(x-y) - \varphi(x)|^p\,\mathrm{d}x. \end{aligned}$$

Condition 2 (mean continuity) of the Kolmogorov compactness criterion (a single element $\{\varphi\}$ is compact!) conclude the proof of (c). The reader may carry out the appropriate alterations to the proof in the case $p=1$ for himself.

Now to (d). Let φ be assumed continuous. Since $\int h_\varepsilon(y)\,\mathrm{d}y = 1$ we can write:

$$\varphi_\varepsilon(x) - \varphi(x) = \int_{y \leqslant \varepsilon} (\varphi(x-y) - \varphi(x))h_\varepsilon(y)\,\mathrm{d}y$$

and the uniform continuity of φ implies the uniform convergence of φ_ε to φ as ε tends to zero. ■

1.4 Distributions

Next we define distributions in the sense of L. Schwartz on Ω. We take $\mathscr{D}(\Omega)$ as the fundamental or test space. We easily see that $\mathscr{D}(\Omega)$ is a linear space, and we can consider linear functionals on $\mathscr{D}(\Omega)$, that is linear maps $T:\mathscr{D}(\Omega)\to\mathbb{C}$. We define:

Definition 1.1 *We call a linear functional* $T:\mathscr{D}(\Omega)\to\mathbb{C}$ *a* distribution, *if for each compact set* $K \subset\subset \Omega$ *we can find constants* C *and* k, *so that the estimate*

$$|T(\varphi)| \leqslant C \sum_{|s|\leqslant k} \sup_K |\mathrm{D}^s\varphi| \quad \textit{holds for all } \varphi\in\mathscr{D}(K). \tag{11}$$

Here the notation $K \subset\subset \Omega$ means that K is compact and $K \subset \Omega$. $\mathscr{D}(K)$ consists of all $\varphi\in\mathscr{D}(\Omega)$ with $\operatorname{supp}\varphi \subseteq K$. We denote the space of all distributions by $\mathscr{D}'(\Omega)$. In the case that we can choose the constant k independent of K, we say that the distribution T is of finite order. The smallest of these ks we call the order of T, and we denote the space of all distributions of finite order by $\mathscr{D}'_F(\Omega)$.

Example 1.1 Let $f\in L_1^{\mathrm{loc}}(\Omega)$. Then

$$T_f(\varphi) := \int f(x)\varphi(x)\,\mathrm{d}x, \quad \varphi\in\mathscr{D}(\Omega), \tag{12}$$

is a distribution of order 0.

Example 1.2 Let m be a Radon measure, that is, $m\in(C_0^0(\Omega))'$ (see Bourbaki [2]). Then

$$T_m(\varphi) := \int \varphi(x)\,\mathrm{d}m, \quad \varphi\in\mathscr{D}(\Omega), \tag{13}$$

is again a distribution of order 0.

Addendum. *The space* $C_0^0(\Omega)$ *is the space of all continuous functions on* Ω *with compact support,* $\operatorname{supp}\varphi \subset\subset \Omega$, *equipped with the inductive limit topology*