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Polytopes and Symmetry

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1. THE SPACE OF POLYTOPES

Polytopes are closely related to two other families of subsets of Euclidean space, namely finite subsets and affine planes. We explore this relationship and construct a natural topology for each of the first two families. The basic equivalence relation of similarity and its interpretation in terms of transformation groups are also discussed.

1. Euclidean space

For any positive integer n, Euclidean n-space will be denoted by E^n. It is convenient to represent E^n as a fixed subspace of E^{n+1}, and we do this by identifying $x \in E^n$ with $Y \in E^{n+1}$, where $y_i = x_i$ for $1 \leq i \leq n$ and $y_{n+1} = 0$. Likewise, we consider the space E of all infinite sequences $x = (x_i) = (x_1, \ldots, x_n, \ldots)$ of real numbers x_i in which $x_i = 0$ for all but finitely many values of i, and we embed E^n in E by identifying $x \in E^n$ with $z \in E$, where $z_i = x_i$ for $1 \leq i \leq n$ and $z_i = 0$ for $i > n$. In this way we obtain a sequence of inclusions

$$E^1 \subset E^2 \subset \ldots \quad \subset E^n \subset E^{n+1} \subset \ldots$$

and we can write $E = \bigcup_{n=1}^{\infty} E^n = \lim\limits_{\rightarrow} E^n$.

The space E has a real linear structure given by

$$\lambda(x_i) + \mu(y_i) = (z_i),$$

where $z_i = \lambda x_i + \mu y_i$, $i = 1, \ldots, n, \ldots$, for $\lambda, \mu \in R$, $x = (x_i)$, $y = (y_i) \in E$. The metrical structure of E is determined by the familiar Euclidean inner product $< , >$, where $<x,y> = \Sigma_{i=1}^{\infty} x_i y_i$ for all $x, y \in E$. The associated norm $\| \;\|$ and Euclidean metric d are then given by $\|x\| = \sqrt{<x,x>}$ and $d(x,y) = \|x - y\|$ respectively.

The resulting metric space (E,d), which we denote simply by E, is called <u>Euclidean space</u>. The fact that E is not complete will cause us no trouble, since we are interested only in subsets of E that lie in some E^n. Our only reason for introducing E is to be able to handle all the spaces E^n at once.

2. Affine hulls of finite sets

Let $\mathcal{F}$ denote the family of all finite subsets of E . We denote the number of elements in $A \in \mathcal{F}$ by $\#A$ (and extend this notation to finite sets in general in Chapter 2). Thus $\#\emptyset = 0$ and $\#A = 1$ iff A is a singleton $\{a\}$, $a \in E$.

If $A \in \mathcal{F}$, then any linear combination $\Sigma_{a\in A} t_a a$ for which $\Sigma_{a\in A} t_a = 1$ is called an <u>affine combination</u> of A . The set of all affine combinations of A is called the <u>affine hull</u> aff A of A , and any such subset $X = \text{aff}A$ of E is called an <u>affine plane</u> generated by A . More generally, for any subset S of E , affS denotes the union of the affine planes affA , where $A \in \mathcal{F}$ and $A \subset S$.

Let $\mathcal{A}$ denote the set of all affine planes of the form affA, $A \in \mathcal{F}$. Then $\text{aff}:\mathcal{F} \to \mathcal{A}$ is a surjective map. For example, $\text{aff}A = A$ iff $A = \emptyset$ or A is a singleton $\{a\}$. Of course it may happen that $\text{aff}A = \text{aff}B$ for distinct sets $A, B \in \mathcal{F}$. If $\text{aff}A = X$ and for every proper subset S of A , aff S is a proper subset of X , then A is said to be <u>affinely independent</u> and to be an <u>affine basis</u> for X.

The set $\mathcal{L}$ of all finite-dimensional linear subspaces of E is a subset of $\mathcal{A}$ with inclusion $\iota:\mathcal{L}\to\mathcal{A}$, say. For convenience, we give $\emptyset$ 'honorary membership' of $\mathcal{L}$, assigning to it the dimension -1 . There is then a parallel projection $p:\mathcal{A}\to\mathcal{L}$ given by $p(X) = L$, where $L = \{x-y:x,y \in X\}$. Trivially, $p\circ\iota = 1_{\mathcal{L}}$. We say that $X, Y \in \mathcal{A}$ are <u>parallel,</u> written $X \| Y$, iff $P(X) \subset P(Y)$ or $P(Y) \subset P(X)$.

The dimension dim L of any $L \in \mathcal{L}$ is well-defined, and we extend this notion to the elements of $\mathcal{A}$ by putting $\dim X = \dim P(X)$ for each $X \in \mathcal{A}$. We say that $X \in \mathcal{A}$ is an <u>affine k-plane</u> iff $\dim X = k$.

Exercise: Let $A \in \mathcal{F}$ and $X = \text{aff}A$. Then $\#A \geq 1 + \dim X$ with equality iff A is affinely independent.

Exercise: Let $A \in \mathcal{F}$. Then $A \subset E^n$ for some n . Hence $\text{aff}A \subset E^n$. Thus if $\mathcal{F}_n$ and $\mathcal{A}_n$ denote the sets of all finite subsets and of all affine planes in E^n , respectively, then $\mathcal{F} = \bigcup_{n\geq 1} \mathcal{F}_n$ and $\mathcal{A} = \bigcup_{n\geq 1} \mathcal{A}_n$.

3. Lattice structure

Both $\mathcal{F}$ and $\mathcal{A}$ inherit the partial order by inclusion from the set 2^E of all subsets of E, and both are lattices with respect to this partial order. In $\mathcal{F}$, the least upper bound and greatest lower bound of $A, B \in \mathcal{F}$ are $A \cup B$ and $A \cap B$ respectively. In $\mathcal{A}$, we denote the least upper bound of X and Y by $X \uplus Y$, while the greatest lower bound of $X, Y \in \mathcal{A}$ is just $X \cap Y$. The fact is that $X \cup Y = X \uplus Y$ iff $X \subset Y$ or $Y \supset X$, since $X \uplus Y$ is the intersection of all affine planes that contain $X \cup Y$.

Exercise: Let $A, B \in \mathcal{F}$ and $X = \text{aff}A$, $Y = \text{aff}B$. Then $A \subset B \Rightarrow X \subset Y$, $\text{aff}(A \cup B) = X \uplus Y$, and $\text{aff}(A \cap B) \subset X \cap Y$. Find A, B such that the last inclusion is proper.

Exercise: For all $L, M \in \mathcal{L}$, $L \uplus M = L + M$. For all $X, Y \in \mathcal{A}$, $p(X \uplus Y) = p(X) + p(Y)$ and $p(X \cap Y) \subset p(X) \cap p(Y)$.

4. Polytopes

Let $A \in \mathcal{F}$. Any affine combination $\Sigma_{a \in A} t_a a$ such that, for all $a \in A$, $t_a \geq 0$ is called a *convex combination* of A, and the set of all convex combinations of A is called the *convex hull* $\text{conv}\, A$ of A. Any subset $P = \text{conv}\, A$, where $A \in \mathcal{F}$, is called a *convex polytope* or simply a *polytope*. We denote the set of all polytopes by $\mathcal{P}$. It follows at once from the definitions that for all $A \in \mathcal{F}$,

$$A \subset \text{conv}\, A \subset \text{aff}A = \text{aff}(\text{conv}A) .$$

If $P = \text{conv}A \in \mathcal{P}$, then P is said to be of *dimension* $\dim P = \dim(\text{aff}A)$, and we refer to P as an *n-polytope* iff $n = \dim P$.

The empty set $\emptyset$ is the unique (-1)-polytope. The 0-polytopes are the singleton subsets of E, and 1-polytopes are closed bounded straight line-segments. A 2-polytope is called a *polygon* and a 3-polytope is called a *polyhedron*.

Notice that if $A, B \in \mathcal{F}$ and $A \subset B$, then $\text{conv}\, A \subset \text{conv}\, B$. A more subtle fact is that if $\text{conv}\, A = \text{conv}\, B = P$, then $P = \text{conv}(A \cap B)$. It follows that for each polytope P there is a unique set $V \in \mathcal{F}$ such that $P = \text{conv}\, V$ and for every proper subset W of V, $\text{conv}\, W$ is a proper subset of P. Thus V is the 'smallest' subset of E whose convex hull is P. The elements of V are called the *vertices* of P , and V is called the <u>vertex set</u> vert P of P . The relationship between vert and conv is shown in the commutative diagram

$$\begin{array}{ccc} & \mathcal{F} & \\ {\scriptstyle \text{vert}}\nearrow & & \searrow{\scriptstyle \text{conv}} \\ \mathcal{P} & \xrightarrow[\;1_{\mathcal{P}}\;]{} & \mathcal{P} \end{array}$$

Thus vert is injective and conv is surjective.

It follows from the first Exercise of §2 that an n-polytope P has at least n + 1 vertices, and has exactly n + 1 vertices iff vert P is affinely independent. Another way of putting this is to say that an n-polytope P has exactly n + 1 vertices iff these vertices are 'in general position' in E . Such an n-polytope is called an n-<u>simplex</u>. Of course for $-1 \leqslant n \leqslant 1$, every n-polytope is an n-simplex. A 2-simplex is called a <u>triangle</u> and a 3-simplex is called a <u>tetrahedron</u>, for reasons that either are familiar already or will become clear shortly.

With the second Exercise of § 2 in mind, we note that for each $P \in \mathcal{P}$ there is a positive integer m such that $P \subset E^m$. Trivially, $m \geqslant \dim P$. We denote the set of all polytopes in E^m by $\mathcal{P}_m$ and note that $\mathcal{P} = \bigcup_{n \geqslant 1} \mathcal{P}_m$.

5. The Hausdorff metric

There is a well-known procedure, due to Hausdorff [Kelley, 1942; Grünbaum, 1967], that can be used here to topologise both $\mathcal{F}$ and $\mathcal{P}$ in a natural way. We describe this first in the general context of metric spaces.

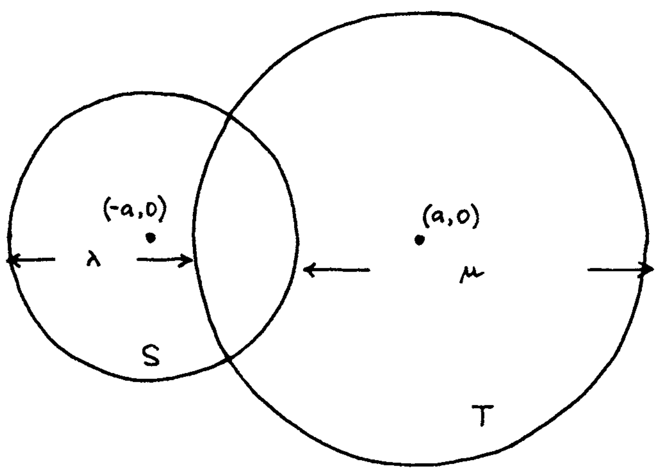
Let (M,δ) be a metric space, and let $\mathcal{K}(M)$ denote the set of all nonempty compact subsets of M . The <u>Hausdorff metric</u> h on $\mathcal{K}(M)$ is defined as follows. Let $S, T \in \mathcal{K}(M)$. Then

$$h(S,T) = \max\{\lambda,\mu\} ,$$

where $\lambda = \max_{x \in S} \min_{y \in T} \delta(x,y)$ and $\mu = \max_{y \in T} \min_{x \in S} \delta(x,y)$. As an illustration, let $M = E^2$, $\delta = d|E^2$ and let S and T be closed circular discs with centres $(-a,0)$, $(a,0)$ and radii r, R respectively, where $0 < r < R < 2a < r + R$. Then $\lambda = r + 2a - R$ and $\mu = R + 2a - r$. Hence in this case $h(S,T) = \mu$, as may be seen readily by reference to Figure 1. It may be shown that h is a complete metric on $\mathcal{b}(M)$, and that every closed bounded subset of $\mathcal{b}(M)$ is compact.

The definition is little more than a formal expression of the commonsense idea that two nonempty compact subsets S and T of M are, in an intuitive sense, 'close' to one another iff each point of S is close to some point of T, and each point of T is close to some point of S.

Figure 1. Hausdorff distance

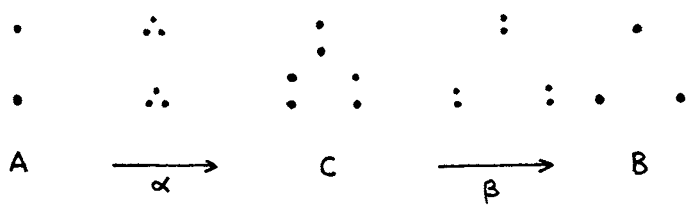


6. The space of polytopes

Let us apply the above construction to the case in which $(M,\delta) = (E,h)$. This yields a complete metric h on the set $\mathcal{K}(E) = \mathcal{K}$ of all nonempty compact subsets of E . If, therefore, we put $\mathcal{F}' = \mathcal{F}\cap\mathcal{K} = \mathcal{F}\setminus\{\emptyset\}$, and $\mathcal{P}' = \mathcal{P}\cap\mathcal{K} = \mathcal{P}\setminus\{\emptyset\}$, then $\mathcal{F}'$ and $\mathcal{P}'$ are subsets of $\mathcal{K}$, and we may give them the induced metrics $h_{\mathcal{F}}$ and $h_{\mathcal{P}}$ respectively and hence the corresponding metric topologies. To each space $\mathcal{F}'$ and $\mathcal{P}'$ we append $\emptyset = \{\emptyset\}$ as a singleton connected component, and so both $\mathcal{F}$ and $\mathcal{P}$ have been topologised.

It is easy to show that each of $\mathcal{F}$ and $\mathcal{P}$ has just two path-components. That is to say, both $\mathcal{F}'$ and $\mathcal{P}'$ are path-connected. For example, let $A, B \in \mathcal{F}'$ with $\# A = r$ and $\# B = s$. Let $C \in \mathcal{F}'$ be any set such that $\# C = rs$. Choose any labellings $a_1, \ldots , a_r \in A$, $b_1, \ldots , b_s \in B$, $c_{11}, c_{12}, \ldots , c_{rs} \in C$ for the elements of A , B and C . For each pair (i,j) $(i=1,\ldots,r;\ j = 1,\ldots,s)$ define a path $\alpha_{ij}:I \to E$ by $\alpha_{ij}(t) = ta_i + (1-t)c_{ij}$, and let $\alpha:I \to \mathcal{F}'$ be given by $\alpha(t) = \{\alpha_{ij}(t): i=1,\ldots,r;\ j=1,\ldots,s\}$, where I denotes the closed interval $[0,1]$. Then α is a continuous path in $\mathcal{F}'$ from C to A . Likewise there is a path β from C to B and hence a path $\beta\alpha^{-1}$ from A to B in $\mathcal{F}'$. Figure 2 illustrates various steps along such a path for $r = 2$, $s = 3$.

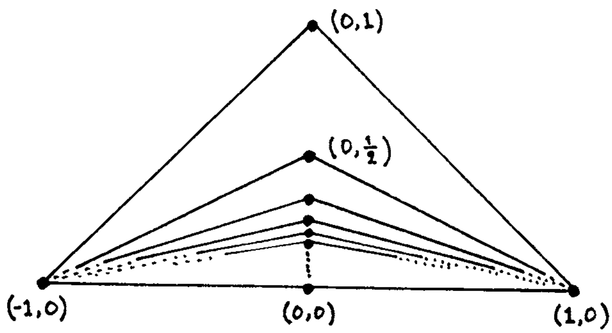
Figure 2. Path-connectedness of $\mathcal{F}'$



It is natural to ask whether the maps vert and conv are continuous with respect to the topologies that we have assigned to $\mathcal{P}$ and $\mathcal{F}$. We observe first that conv is continuous; for if $A, B \in \mathcal{F}'$, $P = \text{conv } A$ and $Q = \text{conv } B$, then $h_{\mathcal{P}}(P,Q) \leq h_{\mathcal{F}}(A,B)$. On the other hand, vert is not continuous. To see this, let us construct a sequence (P_n) of polytopes P_n that converges in $\mathcal{P}$ to P but for which the corresponding sequence (V_n), where $V_n = \text{vert } P_n$, does not converge in $\mathcal{F}$ to $V = \text{vert } P$.

Consider the sequence of triangles P_n, where $\text{vert } P_n = \{(-1,0), (0,{}^1/_n), (1,0)\} \subset E^2$. Then (P_n) converges to the closed line-segment $P = [-1,1]$ in E^1, and so $V = \{-1,1\} = \{(-1,0),(1,0)\}$, while (V_n) converges to $\{(-1,0),(0,0),(1,0)\} = \{-1,0,1\}$. Figure 3 may make this a little more obvious.

Figure 3. vert is not continuous



7. Similarity and congruence

In Euclid's Elements [Heath 1956], two equivalence relations between geometrical figures are much in evidence, either implicitly or explicitly, particularly in Euclid's treatment of triangles. Both relations may be defined in an obvious way on the set of subsets of any metric space, and have the virtue of being definable by means of group actions. Let us look first, then, at similarity and congruence in metric spaces.

Let (M,δ_M) and (N,δ_N) be metric spaces. A map $f:M \to N$ is said to be a similarity iff, for some real number $\lambda > 0$ and for all $x, y \in M$,

$$\lambda\delta_M(x,y) = \delta_N(f(x),f(y)) .$$

The number λ may be called the modulus of f, and we write accordingly $\lambda = |f|$. Let us denote the set of all similarities from M to N by $\mathscr{S}(M,N)$. If $f \in \mathscr{S}(M,N)$ and, for some metric space (W,δ_W), $g \in \mathscr{S}(N,W)$, then $g \circ f \in \mathscr{S}(M,W)$ and $|g \circ f| = |g||f|$. Trivially, $1_M \in \mathscr{S}(M,M)$ and we have, it seems, unwittingly constructed a category $\mathscr{S}$ of similarities between metric spaces. In this category $\mathscr{S}$, we are interested in the group $\mathrm{Aut}_{\mathscr{S}} M$ of invertible self-similarities of (M,δ_M). We call this the similarity group of (M,δ_M) and we prefer to denote it by $\mathrm{Sim}(M,\delta_M)$ or $\mathrm{Sim}\, M$.

The map $\mu:\mathrm{Sim}\, M \to R_*$ into the multiplicative group R_* of positive real numbers, given by $\mu(f) = |f|$, is a homomorphism whose kernel is called the isometry group $\mathrm{Iso}\, M$ of M. The elements of $\mathrm{Iso}\, M$, called isometries, are the metric-preserving bijections of M to itself.

Problems: (i) Let D be a subgroup of R_*. Find a metric space (M,δ_M) such that $\mu(\mathrm{Sim}\, M) = D$. (ii) Let $\Delta = \mathrm{im}\mu < R_*$. Characterise those metric spaces (M,δ_M) for which the short exact sequence

$$1 \longrightarrow \mathrm{Iso}\ M \longrightarrow \mathrm{Sim}\ M \xrightarrow{\mu} \Delta \longrightarrow 1$$

splits. Characterise those spaces (M,δ_M) for which $\Delta = R_*$.

The group $\mathrm{Sim}\, M$ acts on the set 2^M of all subsets of M by $f\cdot X = \{f(x):x \in X\}$, and $\mathrm{Iso}\, M$ also acts on 2^M by restriction of this action. An orbit of $\mathrm{Sim}\, M$ in 2^M is called a similarity class of subsets of M, and an orbit of $\mathrm{Iso}\, M$ is called a congruence class of subsets of M . In alternative language, if $X, Y \in 2^M$, then X is similar to Y , written $X \sim Y$, iff $Y = f \cdot X$ for some $f \in \mathrm{Sim}\, M$, and X is congruent to Y , written $X \equiv Y$, iff $Y = g \cdot X$ for some $g \in \mathrm{Iso}\, M$. Clearly $X \equiv Y$ implies $X \sim Y$, so that congruence is finer than similarity (and similarity is coarser than congruence).

We have approached these relations by using properties of the ambient space M as a whole. Another way to proceed is as follows. For $X \in 2^M$, let δ_X denote the metric induced on X by restriction of δ_M . Then we may choose to define a 'similarity' from (X, δ_X) to (Y, δ_Y) to be an $\mathcal{S}$-isomorphism in the categorical sense discussed above, and to say that X is 'similar' to Y iff such an $\mathcal{S}$-isomorphism exists. If $X \sim Y$, then X is certainly 'similar' to Y in this new sense. It may be impossible, however, to extend a given $\mathcal{S}$-isomorphism from (X, δ_X) to (Y, δ_Y) to obtain an element $f \in \mathrm{Sim}\, M$, so the two approaches do not yield the same concept. Analogous remarks apply to congruence. For more details and references to the background literature, see Robertson [1976].

In the Euclidean spaces that we are studying here, these difficulties do not arise: the two approaches to similarity and congruence lead to the same pair of relations.

8. Euclidean similarity

Let us now consider the case $(M, \delta_M) = (E^n, d)$, where d denotes $d|E^n$. We put $\mathrm{Sim}\, E^n = \mathrm{Sim}(n)$, $\mathrm{Iso}(E^n) = \mathrm{Iso}(n)$, and observe that μ is surjective and the sequence

$$1 \longrightarrow \mathrm{Iso}(n) \longrightarrow \mathrm{Sim}(n) \xrightarrow{\mu} R_* \longrightarrow 1$$

splits (see Problems, §7). Thus $\mathrm{Sim}(n)$ may be identified with a semi-direct product of $\mathrm{Iso}(n)$ with R_* . As a set, therefore, we may identify $\mathrm{Sim}(n)$ with the Cartesian product of the underlying sets of $\mathrm{Iso}(n)$ and R_* . The group structure may be established once we know something about $\mathrm{Iso}(n)$, which itself may be expressed as a semi-direct product.

Let R^n denote real linear n-space, and $O(n)$ the orthogonal group of real $n \times n$ matrices H such that $HH^t = I_n$, where $I_n = [\delta_{ij}]$ is the $n \times n$ unit matrix and the superscript denotes transposition. Then there is a split short exact sequence

$$1 \longrightarrow R^n \longrightarrow \mathrm{Iso}(n) \longrightarrow O(n) \longrightarrow 1$$

under which we may identify Iso(n) as a set with the Cartesian product $R^n \times O(n)$, defining a group operation (juxtaposition) by

$$(a,H)(b,K) = (H\cdot b+a, HK) ,$$

where $\cdot$ denotes the standard action of $O(n)$ on R^n .

We may then identify Sim(n) with the set $R^n \times O(n) \times R_*$ on which a group operation (again indicated by juxtaposition) is defined by

$$(a,H,s)(b,K,t) = (sH\cdot b+a, HK, st) .$$

The action of Sim(n) on E^n is given by

$$(a,H,s)\cdot x = sH\cdot x + a .$$

The next step is to transfer these ideas from E^n to E itself. The inclusion $E^n \subset E^{n+1}$ induces a monomorphism of Sim(n) into Sim(n+1) , sending $(a,H,s) \in \mathrm{Sim}(n)$ to $(a,H',s) \in \mathrm{Sim}(n+1)$, where

$$H' = \begin{bmatrix} H & 0 \\ 0 & 1 \end{bmatrix} .$$

We may construct a group Sim as the direct limit of the resulting sequence of monomorphisms, and we identify Sim with the group constructed as follows. Let I_∞ denote the doubly-infinite matrix whose (i,j)th element is δ_{ij} , and let $\bigcirc$ denote the group of all doubly-infinite matrices of the form

$$\tilde{H} = \begin{bmatrix} H & 0 \\ 0 & I_\infty \end{bmatrix} ,$$

where $H \in O(n)$ for some n . Then $\mathrm{Sim} = E \times \bigcirc \times R_*$, with the group operation given by

$$(a,\tilde{H},s)(b,\tilde{K},t) = (s\tilde{H}\cdot b+a, \tilde{H}\tilde{K}, st)$$

as above, and with $(a,\tilde{H},s)\cdot x = s\tilde{H}\cdot x+a$ for $x \in E$.

The group Sim is a proper subgroup of Sim E . For consider the doubly infinite matrix

$$H_* = \mathrm{diag}(H,H,\ldots,H,\ldots) ,$$

for any $H \in O(n)$. Then H_* acts on E by $H_*\cdot x = y$, where

$$y_{kn+i} = h_{ij}\, x_{kn+j} \quad (k=0,1,\ldots;\ i,j=1,\ldots,n) .$$

Thus $H_* \in \mathrm{Sim}\, E$ but $H_* \notin \mathrm{Sim}$. However, Sim is quite large enough for our purposes. In particular, the action of Sim E on 2^E leaves both $\mathcal{P}$ and $\mathcal{F}$ setwise invariant. Moreover, if $P, Q \in \mathcal{P}$ are $\mathcal{S}$-isomorphic, then there exists $f \in \mathrm{Sim}$ such that $f \cdot P = Q$, and the same applies to $\mathcal{F}$ and to $\mathcal{A}$. In fact, the maps conv, vert and aff are equivariant under the action of Sim.

In analogous fashion, we can construct a group $\mathrm{Iso} < \mathrm{Sim}$ and use this to define congruence in $\mathcal{P}$ and $\mathcal{F}$.

9. The similarity space of polytopes

We are interested in polytopes with regard to their metrical symmetry. From this point of view, what matters is the shape of a polytope, and not its size or its position in space. A theory of symmetry for polytopes, therefore, must be a theory of similarity classes of polytopes rather than of polytopes as individuals. It follows that we should study the quotient space $\mathcal{P}/\sim \; = \mathcal{P}/\mathrm{Sim}$ rather than $\mathcal{P}$ itself.

For reasons that we now try to explain, we give $\mathcal{P}/\sim$ a modified version of the quotient topology. Our aim is to ensure that $\mathcal{P}/\sim$ is a Hausdorff space, retaining as far as possible the natural features of the Hausdorff metric topology on $\mathcal{P}$. The only neighbourhood of the class Θ of singleton subsets in $/\sim$ is $'/\sim$ itself, since any polytope is similar to one that is arbitrarily small. Accordingly, we consider the set $\mathcal{P}'$ as the union of the set $\mathcal{P}^0$ of polytopes of dimension 0 and the set $\mathcal{P}^+$ of polytopes of positive dimension. Thus $\mathcal{P} = \mathcal{P}^+ \cup \mathcal{P}^0 \cup \{\emptyset\}$. We topologise $\mathcal{P}^+/\sim$ by the quotient topology, and attach $\Theta = \mathcal{P}^0/\sim$ and $\emptyset = \{\emptyset\} = \{\emptyset\}/\sim$ as two singleton path-components. The resulting space $\mathfrak{S}$ has three path-components, and is called the <u>similarity space</u> of polytopes. It is convenient to put $\mathfrak{S}^+ = \mathcal{P}^+/\sim$. Thus $\mathfrak{S}^+$ is the similarity space of polytopes of positive dimension.

One of the advantages of proceeding in this seemingly over-pedantic way lies in the fact that every element of $\mathfrak{S}^+$ includes 'normal' representatives, defined in §3.3 below.

We denote the similarity class of $P \in \mathcal{P}$ by $\$P$, and the congruence class of P by $\mathbb{K}P$.