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FPF Ring Theory

Faithful modules and generators of mod-R

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PREFACE

FPF Ring Theory is the study of modules in the category mod-R of all right modules over a ring R, specifically those modules, called generators, which generate the category mod-R, and their relationship to the faithful and/or projective R-modules. Azumaya began the theory when he initiated the study of the algebras that are named after him. This led him to study generators of mod-R (called upper distinguished modules by him) and the first theorems on generators are owed to him.

Morita's seminal and monumental study of the category equivalence between mod-R and mod-S for two rings led him to many generator theorems, especially the classical Morita theorem stating that M generates mod-R iff M is finitely generated projective over its endomorphism ring $B = \text{End } M_R$, and $R \approx \text{End}_B M$ canonically (via right multiplications.)

The condition mod-R $\approx$ mod-S is called Morita Equivalence (M.E.) in his honor, and Morita's Theorem implies that this is right-left symmetric. Thus: mod-R $\approx$ mod-S iff there is a finitely generated projective generator P in mod-R, and a ring isomorphism $S \approx \text{End } P_R$. (When this is so, then $P^* = \text{Hom}_R(P,R)$ is also finitely generated projective as a canonical left R-module, and $S = \text{End}_R P^*$ canonically.)

Azumaya defined the Brauer group Br(k) over any commutative ring k. To define Br(k), consider classes of M.E. algebras, under an operation defined by

$$[A][B] = [A \otimes_R B]$$

for k-algebras A and B. This forms a semigroup S(k) and the identity [k] consists of all A such that A is M.E. to k. Now Br(k) is the group of units of S(k), and actually each [A] ε Br(k) is defined by an Azumaya algebra A!

This book is mainly a study of the associative rings with the property that every finitely generated faithful module is a generator of the category of modules over the ring, called FPF rings. These rings are generalizations of pseudo-Frobenius rings (= every faithful module is a generator) which in turn are generalizations of quasi-Frobenius rings (= self-injective Artinian). This accounts for the name finitely pseudo-Frobenius (FPF). There is, moreover, a connection with the fundamental theorem of abelian groups. Namely, any ring for which each finitely generated faithful module has a free direct summand is FPF. There is, also, a finiteness condition associated with FPF rings not explicit in their name: for all known FPF rings there is a bound on the number of isomorphic one sided ideals in any direct sum contained in the ring. Rings with this property are said to be thin, and they properly include rings with finite Goldie dimension. (An infinite product of commutative self-injective rings, e.g. fields, is thin, indeed FPF, but has infinite Goldie dimension.)

A good deal of the structure of FPF rings is known but many interesting questions remain unanswered. One of the most intriguing is: are all FPF rings thin? (See Open Question for some others.)

In this volume we have organized most of the known facts concerning FPF rings and attempted to make it as self-contained as is practical.

DEDICATION AND ACKNOWLEDGEMENT

The fundamental and pioneering work of Professors Goro Azumaya and Kiiti Morita made possible this systematic study of the relationship between the concepts of "generators" and "faithful modules" of mod-R, and we dedicate this study to them.

We originally entitled this work "Azumaya-Morita Theory" until we realized how much broader than FPF Ring Theory that theory is.

The authors also acknowledge a great and happy debt to Professor Abraham Zaks of the Israel Institute of Technology (TECHNION) for many, many favors, both mathematical and personal. He invited both authors to Haifa, he listened to their lectures, and stimulated them with questions that exhibited his deep understanding of Azumaya-Morita theory. In particular in an early unpublished paper, Faith and Zaks proved that every commutative FPF valuation ring is quotient injective. This proved to be a proto-type theorem for commutative FPF rings. This book is based on the senior author's lecture notes "Faithful Modules and Generators of Mod-R", and he wishes to repeat his thanks given to Mrs. Marks (of Technion) for typing, and Professor John Koehl (Louisiana State University, Baton Rouge) for his critical reading, of the original manuscript. He also gratefully acknowledges Mary Ann Jablonski and "Addie" Bouillé of the Rutgers Mathematics Department staff for many favors and much help.

Many pages of the mathematics of this book were written in that wonderful coffee house in Princeton, PJ's ("A clean, well lighted place" in Hemingway's phrase.) There's no better way to thank a Herbert Tuchman (who loses money everytime one of the authors sits down!) than to tell it here. Thanks Herb, Debbie, Ruby, Alice, Joyce, Willy, Barbara, Joy, Kathy, Hilda, Karen, Liz, Patricia, Kim, Rawl, John, Tony, Maddie, and all.

The authors wish to thank Professor Page's wife Joan Marie for her patience in helping proofread this manuscript as well as his daughters for their general forebearance and finally Mary-Margaret Daisley for her superb job of typing the final manuscript.

Professor Page would like to dedicate this volume to his parents Urlin Scott and Helen Elizabeth Page, his wife Joan Marie and daughters, Marianne Elizabeth and Stephanie Theresa.

Professor Faith gives his share of the book to that talented fourteen-year-old mathematician, Japheth Wood, and to lovely Molly Sullivan.

FPF RING THEORY: FAITHFUL MODULES AND GENERATORS OF MOD-R

INTRODUCTION

In these notes we systematize the study of the property of a ring R, every (finitely generated) faithful module generates the category mod-R of all right R-modules over a ring R. Then the ring is said to be right (F)PF, or (finitely) pseudo-Frobenius.

Since R generates mod-R, and since R is a finitely generated R-module, to say that M generates mod-R is equivalent to saying that there is an onto map $M^n \to R$ of a finite direct sum of copies of M onto R. If we were to let trace(M) denote the trace ideal of M in R and M^* the dual module, then M generates R iff

$$\text{trace}_R(M) = \sum_{f \in M^*} f(M) = R$$

That is, iff there exist finitely many elements $x_i \in M$, $f_i \in M^*$, $i = 1,\cdots,n$, so that

$$\Sigma_{i=1}^{n} f_i(x_i) = 1$$

It is clear for a simple ring R that an R-module M generates mod-R iff trace(M) $\neq 0$, that is, $M^* \neq 0$. However, not just simple rings have this property for example: any prime ring which is Noetherian and for which nonzero ideals are invertible, has this property since, as we shall see, every nonzero right ideal is a generator of mod-R. Furthermore, since every nonzero module is faithful over a simple ring R, then right FPF implies that every simple R-module V embeds in R (via $V^* \neq 0$) and this implies that R is semisimple Artinian. Conversely, any semisimple Artinian ring is 2-sided FPF. Moreover, a Dedekind Prime Ring R is right FPF iff R is right bounded in the sense that a cyclic right module R/I is faithful only if I is an inessential right ideal. This result appears in Chapter 4; that every right FPF ring is right bounded is proved in Chapter 1.

Frobenius algebras were devised as abstractions of group algebras of finite groups over fields, but in generalizing the notion to Artinian rings, Nakayama [39] was led to the more important notion of a quasi-Frobenius ring[1]. Quasi-Frobenius rings are characterized as the Artinian rings in which annihilation defines a duality between the lattices of right and left ideals. Some twenty years later Ikeda [52] characterized quasi-Frobenius rings homologically as right self-injective right Artinian rings. (By the symmetry of quasi-Frobenius rings, they are left Artinian and left self-injective, too.) Quasi-Frobenius rings are PF, for over an Artinian ring the ring R embeds in a finite direct sum of copies of any faithful module, and injectivity of R splits this embedding, giving the desired epimorphism. This argument also works for any commutative self-injective ring and finitely generated faithful module, so all self-injective commutative rings are FPF.

Right PF rings were introduced by Azumaya in 1966 and were characterized by Azumaya, Osofsky and Utumi [66, 66, 67] as right self-injective semi-perfect rings with essential right socles. Nakayama's definition for Artinian Quasi-Frobenius rings called for a pairing of the primitive idempotents $e_i \to e_i'$ so that the top of e_iR is isomorphic to the bottom of $e_i'R$ and similarly for Re_i and Re_i', $i = 1,\dots,n$. By a result of Faith [66] any right self-injective ring with ascending or descending chain condition on right annihilators is quasi-Frobenius, so all one sided Noetherian or Artinian PF rings are quasi-Frobenius. Osofsky [66] gave an example of a non-Artinian PF ring. Others were constructed in Faith [79a] as split-null

[1]Nakayama [39] refers to a paper, book, or work by Nakayama listed in References. If more than one should appear, then [39a] would refer to the first, and [39b] to the second, etc.

extensions $R = (B,E)$ of a (B,B)-bimodule E over a commutative ring B, where $(B,E) = \{\begin{pmatrix} b & x \\ 0 & b \end{pmatrix} \mid b \in B, x \in E\}$. Then $R = (B,E)$ is injective iff E is injective and $B = \mathrm{End}_B(E)$ canonically. Moreover, then $R = (B,E)$ is PF iff E is an injective cogenerator over B. This yields a plentiful supply of PF and FPF rings. (See Chapter 5 for these examples.)

While Osofsky's construction of a PF ring was made with the aim of showing that PF does not imply QF, nevertheless PF rings are semiperfect, which is to say semilocal rings with idempotents lifting modulo the Jacobson radical, whereas the Product Theorem for FPF Rings of Faith [79c] implies that any product of commutative, or self-basic (self-basic means semiperfect and modulo the radical a product of skew fields) FPF rings is FPF. Thus, for example, for any cardinal α, and FPF ring R, the product R^α is FPF, e.g. $\mathbb{Z}^\omega$, $\mathbb{Q}^\omega$, $\mathbb{R}^\omega$, $\mathbb{H}^\omega$, or R^ω for any field R. This provides a powerful incentive for the study of FPF rings. Not only is $\mathbb{Z}^\omega$ FPF, but R^α for <u>any</u> commutative or self-basic ring FPF ring R; for example if R is any self-basic PF ring then R^α is FPF but not PF for $\alpha \geq \aleph_0$. This subject is taken up in Chapter 1, more generally for "generic" families. (See Theorem 1.22A ff.)

Commutative FPF rings are characterized* by the two properties:

(FPF 1) Every finitely generated faithful ideal is projective.

(FPF 2) R has injective quotient ring $Q_c(R)$.

When (FPF 2) holds, R is said to be <u>quotient-injective</u>. This theorem is illustrated by the result that a domain R is FPF iff R is Prufer. (In view of the fact that a commutative self-injective ring R is FPF, the theorem indicates that finitely generated faithful ideals of a self-injective ring R are projective, but the fact is that R is the only one!)

A commutative ring R is FGC if every finitely generated module is a direct sum of cyclic modules,

---

*The results on commutative FPF rings are, for the most part, not taken up in these notes, and may be found in Faith [79a, 82a] that is to say, the basis theorem for abelian groups extends to FGC rings.

It is easy to see that <u>any FGC ring is FPF</u>: if M is finitely generated, then $M = R/I_1 \oplus \cdots \oplus R/I_n$ for ideals $I_1 \subset I_2 \subset \cdots \subset I_n$ (in accordance with the structure theory of FGC rings [see e.g. Brandal [79] or Vamos [77]]), so M is faithful only if $I_1 = 0$, so $M \approx R \oplus X$ in mod-R as required.

Trivially, any factor ring of an FGC ring is FGC, so <u>any factor ring of an FGC ring is FPF</u>. We call the latter property CFPF, that is, a ring R is right CFPF if every factor ring is right FPF.

A ring R is said to be <u>linearly compact</u> provided that all systems

$$x \equiv x_i \bmod I_i$$

of congruences defined by ideals $\{I_i\}_{i\in A}$ and elements $\{x_i\}_{i\in A}$ of R are solvable if every finite subsystem is. A valuation ring R (= a chain ring, or a ring with linearly ordered ideal lattice) is said to be <u>maximal</u> provided that R is l.c., and <u>almost maximal</u> if R/I is l.c. for all ideals $I \neq 0$.

Now Kaplansky [49] proved that all almost maximal valuation rings (AMVR's) are FGC, and later raised the problem of constructing all FGC rings. The solution to the problem appears in Brandal [79], Vamos [77], and Wiegands [77]. Kaplansky [42] constructed rings of formal power series $\Sigma_{\gamma\varepsilon\Gamma}\alpha_\gamma x^\gamma$ in a variable x, with coefficients α_γ in a field, and exponents γ coming from a totally ordered group Γ, and showed these rings are MVR's, i.e. there exist MVR's with arbitrary value group Γ. Thus the MVR's form an important class of CFPF rings; in fact, a local ring R is CFPF iff R is an AMVR (see, e.g. [Faith 79a]).

In 1969, Tachikawa [69] proved that a left perfect right FPF ring R is right PF, hence any right or left Artinian right FPF ring is QF. This inspired the following result of Faith [77]: if R is a semiperfect right FPF ring with nil radical, then R is right self-injective. Then Tachikawa's theorem follows from the Azumaya-Osofsky-Utumi theorem. In Chapter 2 we study semi-perfect right FPF rings. If R is semiperfect right FPF, then $R = \Sigma_{i=1}^{n} e_iR$, where $e_i^2 = e_i$, $1 = \Sigma_{i=1}^{n} e_i$ and e_iR is a uniform right ideal, $i = 1,\ldots,n$. Moreover, the basic ring R_0 of R is strongly right bounded in the sense that every nonzero right right ideal contains a nonzero ideal. More can be said if the radical J of R is nil, for then R is right self-injective. (A partial converse proved in Chapter 2: Any right self-injective semiperfect ring R with strongly right bounded R_0 is necessarily right FPF.) Also, if the left zero divisors are right zero divisors for a right FPF semiperfect ring it is shown that the maximal right quotient ring is right injective and is the left classical quotient ring. This covers all known semiperfect right FPF rings.

To continue: Semiprime semiperfect right FPF rings are semihereditary and finite products of full matrix rings of finite rank over right bounded local Ore domains which are right and left valuation rings. It is shown that the basic ring of a semiperfect right CFPF ring is right duo (right ideals are two sided), right σ-cyclic (finitely generated modules are direct sums of cyclics) and finite products of right valuation rings.

Morita [58] characterized the situation when there exists a functor duality D between certain subcategories of mod-R containing R and all finitely generated R-modules, and corresponding subcategories of mod-S, for some ring S. This happens iff there exists an (S,R)-bimodule U which is an injective generator both in the category mod-R of all right R-modules and the corresponding category S-mod of left S-modules. Moreover, it is required that $R = \mathrm{End}_S U$ and $S = \mathrm{End}\, U_R$. Then R is said to possess a Morita duality, and the contravariant functor $h_U = \mathrm{Hom}_R(\ ,U)$ induces the duality D, and $\mathrm{Hom}_S(\ ,U)$ induces D^{-1}. The symbol ${}_SU_R$ is then called a Morita duality context. (Consult ART, Chapter 23 for further details.*)

The connection with PF rings is this: R is right PF iff R is an injective cogenerator in mod-R. Therefore, ${}_RR_R$ is a Morita duality context iff R is right and left PF.

Camillo-Fuller [76] characterize right (F)PF by the condition that (finitely generated) faithful right R-modules are flat as modules over their endomorphism rings. (The FPF part is implicit in their paper. But see corollary 1.19.)

On the subject of generators, a theorem of Morita [58] states that any generator M of mod-R has the property that M is finitely generated projective over its endomorphism ring. Therefore by the theorem of Camillo-Fuller, flatness-over-endomorphism ring for all (finitely generated) faithful right R-modules is equivalent to this stronger property.

By a theorem of Gabriel, if C is an abelian category with enough injectives, and if S is the left adjoint to $T:C \to D$, where D is abelian, then T preserves injectives iff S is exact. Thus, M is flat over its endomorphism ring E iff

$$\mathrm{Hom}_R(M,\ ) : \text{mod-}R \rightarrowtail \text{mod-}E$$

preserves injectives. (See Gabriel [62]; also Theorem 6.29 and its corollary in ARMC. Incredibly, the latter is related to a theorem of Bourbaki-Lambek, loc. cit. p. 28.)

In 1967, Endo [67] proved that a Noetherian commutative ring R is FPF (=(FG) in Endo's terminology) iff R is a finite product of Dedekind domains and QF local rings. Moreover, Endo also determined all right FPF rings R which are projective A-orders in a semisimple K-algebra Σ, where A is a Noetherian domain with quotient field K: R is a hereditary, maximal order in Σ. Thus right FPF implies left FPF in this case.

*ART refers to Faith [76]

In chapter 2 we show all Noetherian semiperfect FPF rings are orders in Quasi Frobenius rings. Then, using the results in Chapters 3 and 4, we show in Chapter 5 they are actually products of bounded Dedekind rings and Quasi Frobenius rings. Moreover, Endo also studied the situation where every finitely generated projective faithful R-module generates mod-R. In Chapter 4 we generalize these results and study semi-prime FPF rings satisfying other related finite conditions.

A ring R is right nonsingular if the right annihilator of each nonzero element is not essential. The maximal right quotient ring of such a ring is always a right self-injective von Neumann regular ring. (A ring is von Neumann regular if every module over it is flat.) As we have seen, not every FPF ring is self-injective; but any nonsingular PF ring is, in fact, semi-simple Artinian since the singular ideal is the Jacobson radical for self-injective rings. In Chapter 3 we study the nonsingular FPF rings. It is shown that the regular right FPF rings are characterized as the self-injective regular rings of bounded index (on the index of nilpotent elements). This says that for regular rings right FPF implies left FPF, which is not true in general (see Chapter 5). This characterization enables one to show that for right nonsingular right FPF rings the right and left maximal quotient rings coincide. Since right nonsingular right FPF rings are semiprime (and conversely) one obtains the fact that a right nonsingular right Goldie FPF ring is also left Goldie.

For modules N and M over a ring we say M has N-width a if a is the largest cardinal number such that a direct sum of a copies of N embeds in M. If for M there is a finite number ℓ such that the N-width of M is less than ℓ for all $N \neq 0$ then we say M is thin, and otherwise M is thick. A ring is right thin if it is thin as a right module. Commutative self-injective rings, and, of course Goldie rings are thin, whereas full linear rings on infinite dimensional spaces are thick. All rings with thin regular maximal quotient rings are right thin, hence every nonsingular-right-FPF-ring-is-right-thin. In Chapter 5 we explore this concept and show that a self-injective ring is right FPF iff it is thin and its "basic ring" is strongly bounded. Thin self-injective rings have a "basic ring" much as do semiperfect rings. This allows one to parallel the theory of semiperfect FPF and CFPF rings. This is taken up in Chapter 5 where we also consider FPF group rings. (In general, a finite group G does not yield an FPF group ring RG over an FPF ring R, e.g. $\mathbb{Z}G$ is never FPF.)

All known (right) FPF rings R are (right) thin, and right quotient-injective in the sense that the classical right quotient ring $Q = Q_c^r(R)$ exists and is right injective. The problem of determining whether all right FPF rings are thin and right quotient-injective and other problems related to the structure of FPF, is appended at the end of the text.

A number of results stated above hold in the context of right $(F)P^2F$ rings, or rings over which every (finitely presented) faithful right R-module generates mod-R. A right $C(F)P^2F$ ring is one which is $(F)P^2F$ modulo any ideal. Any valuation ring R is CFP^2F, and this property characterizes VR's among local rings, in analogy with the theorem which shows that CFPF characterizes AMVR's among VR's.