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978-0-521-27738-9 - FPF Ring Theory: Faithful Modules and Generators of Mod-R
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Excerpt
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1 THE BASICS

This chapter provides a format for the statements of a number of key theorems used repeatedly in the sequel, especially theorems from noncommutative ring theory which are used in Chapters 2, 3, and 5. Naturally there is a limit to what can be fitted into such a format--boredom, if nothing else, would limit any list of needed theorems--so certainly many useful theorems are relegated to the status of ad hoc citation. What follows therefore are the basics (or what has been called the bare bones!).

Because of the frequency of the references, we will abbreviate the two main references as follows:

ARMC denotes Algebra: Rings, Modules, and Categories, I.

ART denotes Algebra II: Ring Theory.

1.1A DEFINITION AND PROPOSITION.

Let mod-R denote the category of right R-modules for a ring R. An object M of mod-R is a generator iff the equivalent conditions hold:

G1. *The set-valued functor* $\mathrm{Hom}_R(M, \ )$ *is faithful.*

G2. *Given an object* X *of mod-R, there is an index set* I *and an exact sequence* $M^{(I)} \to X \to 0$, *where* $M^{(I)}$ *is the co-product (= direct sum) of* I *copies of* M.

G3. *There is a finite integer* $n > 0$, *an object* Y *of mod-R, and an isomorphism* $M^n \approx R \oplus Y$.

G4. *The trace ideal is the unit ideal, that is,*

$$\mathrm{trace}_R M = \sum_{f \in M^*} f(M) = R, \quad \text{where} \quad M^* = \mathrm{Hom}_R(M,R).$$

1.1B DEFINITION.

A ring is said to be right FPF (FP^2F) if every finitely generated (presented) faithful right module is a generator. The ring is right CFPF (CFP^2F) if every homomorphic image is right FPF (CFP^2F). PF and CPF rings are defined similarly. see 1.7a

1.1C DEFINITION AND PROPOSITION (THE MORITA THEOREM).

Let R-mod denote the left-right symmetry of mod-R. Two rings A and B are similar, or Morita equivalent, written $A \sim B$, provided that the equivalent conditions hold:

S1. mod-A ≈ mod-B.

S2. There exists a finitely generated projective generator P of mod-A such that $B \approx \text{End } P_A$.

S3. A-mod ≈ B-mod.

In the case S2, $\text{Hom}_A(P, \ )$ induces an equivalence mod-A ≈ mod-B and the left adjoint $\otimes_B P$ is the inverse equivalence. (The equivalence of S1-S3 is Morita's theorem [58]. Cf., Bass [62,68] or ARMC Theorem 4.29.) Also, ideals of A correspond to ideals of B in such a way that $A/I \sim B/I'$, where I' is the ideal of B corresponding to I. (See ARMC, p. 219, 4.31.3).

1.1D THEOREM (Morita).

A right R-module M generates mod-R iff M is f.g. projective over $B = \text{End } M_R$ and $R = \text{End}_B M$ canonically.

The proof of 1.1C is a bit of linear algebra. (See for example, ARMC, p.327, Prop. 7.3).

1.2A KRULL-SCHMIDT THEOREM AND EXCHANGE LEMMA.

Let

$$(1) \qquad M_1 \oplus \cdots \oplus M_n = A \oplus B$$

be a decomposition in R-mod such that $\text{End } A_R$ is a local ring. Then, there exists i, $1 \leq i \leq n$, and an isomorphism $M_i \approx A \oplus X$ for some $X \in$ mod-R. In particular, if M_i is an indecomposable module, $i = 1,\ldots,n$, then (1) implies that $A \approx M_i$ for some i.

Let

(2) $$M = M_1 \oplus \cdots \oplus M_n = N_1 \oplus \cdots \oplus N_m$$

be two decompositions of a module M into direct sums of modules M_i and N_j each with local endomorphism rings, for each i and j. Then, $m = n$, and there is an automorphism f of M and a permutation p on n symbols such that $f(M_i) = N_{p(i)}$ $i = 1,\ldots,n$.

Furthermore, if $M = A \oplus B$, then A can be decomposed into a direct sum of modules each with local endormorphism ring.

Refer to Bass [68], or ART, pp. 39-40.

SEMIPERFECT RINGS

Let $R = \bigoplus_{i=1}^{n} e_i R$ be a direct sum decomposition of R into principal indecomposable right ideals $e_1R,\ldots,e_nR$, where e_iRe_i is a local ring, $i = 1,\ldots,n$. By definition, then, e_i is an idempotent $\neq 0$, and e_iR is an indecomposable right ideal, which we call a right prindec, for short, $i = 1,\ldots,n$. By a theorem of Bass [60], a ring R has such a decomposition if (and only if) R is semiperfect in the sense that $R/\text{rad } R$ is semi-simple, or, as we say, R is semilocal, and idempotents of $R/\text{rad } R$ lift. (Consult Chapters 18 and 20 of ART.)

THE BASIC MODULE AND BASIC RING

Now assume the notation above. Renumber idempotents if necessary so that $e_1R/e_1J,\ldots,e_mR/e_mJ$ constitute the isomorphism classes of simple right R-modules. Thus, every simple module is isomorphic to some e_iR/e_iJ with $i \leq m$ and $e_iR/e_iJ \approx e_kR/e_kJ$ iff $i = k$, for all i and $k \leq m$. The right ideal $B = e_1R + \cdots + e_mR$ is called the basic (right module of R, $e_0 = e_1 + \cdots + e_m$ is then called the basic idempotent, and $e_0Re_0 \approx \text{End } B_R$ is the basic ring of R. The basic module is unique up to isomorphism, and if f_0 is any other basic idempotent, there is a unit x of R such that $f_0 = xe_0x^{-1}$.

A projective module P is a generator iff every simple right R-module is an epic image of P (ARMC, p.148). Thus the basic module B of R is a finitely generated projective generator of mod-R, and hence, by the Morita theorem R is similar to its basic ring $R_0 = \text{End } B_R$.

A semiperfect ring R is selfbasic if R = B, in which case $R = R_0$. This condition is right-left symmetric, inasmuch as R is selfbasic if R/rad R is a product of division rings. The basic ring of R is selfbasic. The basic ring R_0 is also right-left symmetric, that is, the left basic ring $\approx R_0$.

The basic ring is a finite product of local rings iff R is a finite product of full matrix rings over local rings. In this case, R_0 is said to be local-decomposable, and R is said to be primary-decomposable. (see, for example, ART, pp.44-50).

1.2B THEOREM.

Let R be a semiperfect ring with basic right module B, and basic ring R_0. Then, R is similar to R_0. A module M generates mod-R iff B is isomorphic to a direct summand of M. Thus, if R is selfbasic[1]*, then M generates mod-R iff $M \approx R \oplus Y$ in mod-R.*

Proof. As stated above, $R \sim R_0$ and B is a generator of mod-R, and hence so is any module containing B as a direct summand. Conversely, by Theorem 1.1A, a right module M generates mod-R iff R is isomorphic to a direct summand of M^n, for some n, and since B is a direct summand of R, we must have $M^n \approx B \oplus Y$ in mod-R. However since B is a direct sum of indecomposable modules e_iR with local endomorphism rings e_iRe_i, and $e_iR \not\approx e_jR$, $i \neq j = 1,\dots,m$, then by the Exchange Lemma 1.2A, each e_iR

1. For the case when R has finite Goldie dimension, see Corollary 1.12B.

is isomorphic to a direct summand of M, and by repeated application of the lemma, B is isomorphic to a direct summand of M.

1.2C COROLLARY.

Under the same assumptions, an epic image M/I *of a module* M *generates* mod-R *iff* $M = B' \oplus C$ *such that* $I \subset C$ *and* $B' \approx B$. Then, $M/I \approx B \oplus C/I$.

An epic image B/I *of* B *generates* mod-R *iff* $I = 0$. *Thus, if* R *is selfbasic, then a cyclic module* R/I *generates* mod-R *iff* $I = 0$.

Proof. By 1.2B, $M/I \approx B \oplus X$ in mod-R. This means that there are submodules A and C such that $M = A + C$ and $I = A \cap C$ and $A/I = B$. By projectivity of B, I splits in A, so $A = I \oplus B'$ in mod-R, and since $I \subseteq C$, then $M = B' + C$. Since $B' \cap C \subseteq B' \cap A \cap C \subseteq K \cap I = 0$, then $M = B' \oplus C$. But $B' \approx A/I \approx B$, and hence $M/I \approx B' \oplus C/I \approx B \oplus C/I$.

By the Krull-Schmidt, or unique decomposition theorem, $B = B' \oplus C$, with $B' \approx B$ only if $C = 0$, so B/I generates mod-R only if $I = 0$. The last statement is immediate.

The next theorem states that FPF (FP^2F) are *Morita invariant* properties.

1.2D THEOREM.

A ring R *is right* FPF (FP^2F) *iff every ring* A *similar to* R *is right* FPF (FP^2F).

Proof. Let S:mod-R ≈ mod-A be an equivalence. As stated in 1.1B, the ideals of R and A are in a correspondence $I \leftrightarrow I'$ such that $R/I \sim A/I'$, and it follows that for any R-module M, if $I = \text{ann}_R M$, then $I' = \text{ann}_A SM$; hence M is faithful iff SM is faithful. Similarly, f.p., f.g., etc. are categorical or Morita invariant properties. (See e.g. ARMC, Chapter 2, p.92). For f.p., all that is needed is that f.g. is Morita invariant, since S preserves quotients. But a module M is f.g. iff its proper subobjects form an inductive set (ARMC, p.125, 3.8). Thus, since proper submodules are Morita invariant (ARMC, p.92, 2.4(3)) then so are f.g. modules.

1.2E THEOREM.

The properties (C)FPF and (C)FP^2F are Morita invariant.

Proof. This follows from 1.2C, and the fact that under the correspondence $I \to I'$ for ideals, that $R/I \sim A/I'$.

A ring R is right duo, or invariant, provided that every right ideal is an ideal. This appears to be a non-trivial and useful concept, e.g. right Noetherian right VR's are right duo and every right VR with exactly one prime ideal $\neq$ R is right duo (Brungs [69] and Brungs-Torner [77], resp.). Also, in his Theory of Rings, Surveys of the Amer. Math. Soc. (1943) Jacobson proved that every right and left PID is duo. Duo rings are related to regular FPF rings and semiperfect FPF rings in Chapter 3, and also in Proposition 1.5 following. Related to the duo rings are the bounded rings.

BOUNDED RINGS

1.3A DEFINITIONS.

A ring is right bounded if every essential right ideal contains a non-trivial two sided ideal. It is right strongly bounded if every nonzero right ideal contains a nonzero two sided ideal and fully right bounded if for every prime ideal the factor ring is right bounded.

1.3B PROPOSITION. (Faith [76c, 77])

Any right FPF ring R is right bounded.

Proof. Let I be any essential right ideal. If R/I is faithful, then there exists an integer $n > 0$, R-module X, and an isomorphism

$$h : (R/I)^n \longrightarrow R \oplus X.$$

Let $x_1,\dots,x_n \in R$ be such that $h([x_1+I],\dots,[x_n+I]) = 1$, where $1 \in R \subset R \oplus X$. If $x^{-1}I = \{a \in R \mid xa \in I\}$, then we have $\ker h = \bigcap_{i=1}^{n} x_i^{-1}I = 0$. However, $x^{-1}I$ is an essential right ideal for any $x \in R$. To see this, let $Q \neq 0$ be a right ideal. Then $xQ = 0 \to Q \subset x^{-1}I \cap Q \neq 0$. On the other hand, $xQ \neq 0$ means $I \cap xQ \neq 0$, so there is an element $y = xq \neq 0$ in $I \cap xQ$, and then $0 \neq q \in x^{-1}I \subset Q$. This contradiction proves the proposition.

The next proposition will be used in several instances in the sequel.

1.3C PROPOSITION.

If $R = \prod_{i=1}^{n} R_i$ is a finite product of right bounded rings, then R is right bounded. The converse fails. However, if R is strongly right bounded, then so is each $R_i, i = 1,\ldots,n$.

Proof. If I is any essential right ideal of say R, then $R_i \cap I, i = 1,\ldots,n$, is an essential right ideal of R_i, and hence contains an ideal $A_k \neq 0$ of $R_i, i = 1,\ldots,n$, and hence I contains the ideal $A = A_1 + \cdots + A_n \neq 0$.

In the opposite direction, let $R = A \times B$ be a product of an arbitrary ring A and a field B, and let I be any essential right ideal. Then $I \cap B \neq 0$, and hence $I \cap B = B$ is an ideal of R contained in I. This proves that R is right bounded, even if A is not.

If R is strongly right bounded, then any right ideal I of R_i must contain an ideal $\neq 0$, hence R_i is strongly right bounded.

1.3D NOTE

Strongly right bounded implies that any right ideal $I \neq 0$ contains an ideal A which is essential in I.

Proof. For if A is the sum of the ideals contained in I, and if $K \cap A = 0$ for some right ideal K contained in I, then $K \neq 0$ would imply that K contains an ideal $\neq 0$, contradiction, hence $A \cap K \neq 0$.

An object M of mod-R is said to be compactly faithful provided that R embeds in M^n, for a finite integer $n > 0$. (In general, a module M is faithful iff R embeds in a direct product of copies of M.) Thus, by 1.1A, every generator is compactly faithful. A ring R is right

Artinian iff every module M in mod-R is compactly faithful in mod-R/A, where $A = \mathrm{ann}_R M$. (See, for example, ART, pp. 67-69).

For a set X we will let $X^{\perp}({}^{\perp}X)$ *denote the right(left) annhilator of X .*

1.3E *PROPOSITION*

A finitely generated faithful module M over a right strongly bounded ring R is compactly faithful.

Proof. Write $M = \Sigma_{i=1}^{n} b_i R$ for elements $b_1,\ldots,b_n$ in M. Now R strongly right bounded, and M faithful, means that $\bigcap_{i=1}^{n} b_i^{\perp} = 0$, and hence $a \mapsto (b_1 a,\ldots,b_n a)$ is the desired embedding of R in M^n .

1.4 COROLLARY

Any right selfinjective strongly right bounded ring R is right FPF. (*Over a right selfinjective ring R, a compactly faithful module is a generator*). *Thus, any semiperfect right selfinjective ring R with strongly right bounded* basic ring is right FPF.*

Proof. Let M be finitely generated and faithful. By 1.3, R embeds in M^n, and then injectivity of R implies that R is a summand of M^n, so M is a generator by 1.1A. The last statement follows from the fact that R is right selfinjective iff the basic ring R_0 is. Then, apply the first statement, and Theorem 1.2D.

1.4 shows that a right FPF ring need not be semiperfect since any product of right selfinjective commutative rings will be right selfinjective and duo, hence FPF. However, an infinite such product cannot be semilocal. (Also, Z is FPF but not semilocal!)

A ring R is said to be *completely right selfinjective* if every factor ring is right selfinjective.

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*Any semiperfect right FPF ring has a strongly bounded basic ring (see Theorem 2.1) Any right duo, hence any strongly regular ring, is strongly right bounded. Moreover, any right nonsingular FPF ring R has a right quotient ring $Q_{c\ell}(R)$ that is Morita equivalent to a strongly regular ring.

1.5 COROLLARY

Any completely right selfinjective right duo ring R is right CFPF.

Proof. Any factor ring of a right duo ring is right duo, and every right duo ring is strongly right bounded, so 1.4 applies.

Levy [66] gave an example of a non-Noetherian commutative ring R of which all factor rings modulo nonzero ideals are selfinjective rings, and some of the factor rings are PF. The ring exhibited is the ring R of all formal power series in a variable x indexed by the family W of all well-ordered sets of nonnegative real numbers. Thus, an element r of R has the form $r = \Sigma_{i \varepsilon W} a_i x^i$, with $a_i \varepsilon R$, and unique $i \varepsilon W$. The only nonzero ideals of R are: the principal ideals (x^b), and those ideals of the form

$$(x^{>b}) = \{x^c u \mid c > b \text{ , and } u \text{ a unit of } R\} .$$

Thus, if I is any nonzero ideal, then $\bar{R} = R/I$ is completely selfinjective (and non-Noetherian). [Note, however, that we are not asserting that every cyclic $\bar{R}$-module C is injective as an $\bar{R}$-module, but merely injective as an $\bar{R}/A$-module, where $A = \text{ann}_{\bar{R}} C$. Nevertheless, C is quasi-injective as an $\bar{R}$-module. If every cyclic $\bar{R}$-module were injective as an $\bar{R}$-module, then $\bar{R}$ would be a semisimple ring by Osofsky's theorem [64]. This is not the case, since $\bar{R}$ is non-Noetherian.] Osofsky [66] gave some other examples. (See Chapter 5 for some general constructions of self-injective rings and Kaplansky [42] for more general almost maximal valuations constructed as formal power series $\Sigma_{\gamma \varepsilon \Gamma} a_\gamma x^\gamma$ where Γ is a totally ordered abelian group.)

SERIAL AND QF RINGS

A ring R is said to be right serial provided that R is semiperfect and the set of submodules of every right prindec is linearly ordered. A right valuation ring (VR) is a right serial local ring. A right and left serial ring is said to be _serial_. An Artinian serial ring has the property that every right or left module is a direct sum of cyclic modules each of which are homomorphic images of prindecs (Nakayama [40]). A ring R is said to be (right) Σ-_cyclic_ if every (right) module decomposes into a direct sum of cyclic modules. A ring is (right) σ-_cyclic_ if this holds for all finitely generated modules. In particular, Artinian serial rings are Σ- cyclic. (Remark: any right Σ-cyclic ring has to be right Artinian etc.; see Chapter 20 of ART for references to this.) Warfield [75] proved that Noetherian serial rings are σ-cyclic, in fact, every indecomposable cyclic is an epic image of a prindec. A _primary-decomposable serial ring_, or _uniserial_ ring, is a finite product of matrix rings over local serial rings. Asano [49] characterized Artinian uniserial rings as (right and left) Artinian (right and left) principal ideal rings.

1.6 THEOREM

A ring R _is Quasi-frobenius_ (QF) _in case_ R _has the equivalent properties_:

QF(a). _Every right ideal, and every left ideal, is the annihilator of a finite subset of_ R.

QF(b). _Every right ideal, and every left ideal, is an annihilator_ (= _annulet_), _and_ R _is right or left Artinian or Noetherian_.

QF(c). R _is right selfinjective, and right or left Artinian or Noetherian_.

For a discussion, see, e.g., Faith [66], ART, Chapt. 24 (Note the condition QF is left-right symmetric.)

Some relationships between the various rings are: A right Artinian ring is uniserial iff every factor ring is QF. An Artinian ring R is serial iff R/J^2 is QF. For these results, see Nakayama [39, 40, 41], or ART, Chaps. 24 and 25.

The ring of lower triangular matricies $T_n(R)$ over a semisimple ring R is serial, and the injective hull of the right regular module is the full $n \times n$ matrix ring R_n. Thus, $T_n(R)$ is not selfinjective, hence not QF, hence not