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An F-space sampler

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To Adrian and Verona Roberts

To Jennifer

To Emily

CONTENTS

PREFACE

Typically, a basic text on functional analysis will only make the briefest of references to general topological vector spaces, before restricting attention to the locally convex case or to Banach spaces. Thus most analysts are aware of the existence of non-locally convex spaces such as $L_p(0,1)$ for $0 < p < 1$ but know very little about them. The neglect of non-locally convex spaces is easily understood. The basic theory of Banach spaces, which sits at the core of modern functional analysis, may be said to depend on two major principles - the Hahn-Banach theorem and the Closed Graph theorem (which may be taken to include weaker theorems such as the Uniform Boundedness Principle). Working in non-locally convex spaces, even when they are complete and metrizable, requires doing without the Hahn-Banach theorem. The role of the Hahn-Banach theorem may be said to be that of a universal simplifier - infinite-dimensional arguments can be reduced to the scalar case by the use of the ubiquitous linear functional. Thus the problem with non-locally convex spaces is that of "getting off the ground." This difficulty in even making the simplest initial steps has led some to regard non-locally convex spaces as simply uninteresting. It's our contention, which we hope to justify in these notes, that this attitude is mistaken and that with the aid of fresh techniques one can develop a rich and fulfilling theory.

Our aim, therefore, in these notes is to present some aspects of the theory of F-spaces (complete metric linear spaces) which we hope the reader will find attractive. We do not aim to be encyclopaedic, nor do we strive for complete generality in the results which we present. The account is intended to be reasonably self-contained, at least for the reader versed in the basics of topological vector space theory (see e.g. Rudin [1973] or Kothe [1969]). For more background one should refer to Rolewicz [1972] which gives a fairly complete summary of the state of the art up to 1972.

In selecting the material for these notes we have adopted the theme of taking certain familiar properties or theorems from Banach space theory and examining their behavior in general F-spaces. Thus we shall consider in detail the fate of the Hahn-Banach theorem and the Krein-Milman theorem. Also the study of compact operators in a non-locally convex setting takes on a new twist, largely because much of the Fredholm theory can be extended. We place special emphasis on the classical examples of non-locally convex F-spaces - the sequence spaces ℓ_p $(0 < p < 1)$, the function spaces L_p $(0 \leq p < 1)$ and the Hardy spaces H^p $(0 < p < 1)$.

We now turn to a summary of the contents chapter by chapter.

Chapter 1: Preliminaries. This chapter, after the introduction, recalls some of the basic properties of F-spaces. Many readers will be familiar with the contents of this chapter, but a brief perusal is advisable if only to establish notation. We treat the problem of determining an invariant metric on a metric linear space from a slightly unusual point of view by introducing the idea of a Δ-norm. The Closed Graph and Open Mapping theorems are given in a particular form, which is perhaps not as well-known as it should be, since this form is required later in the book.

Chapter 2: Some of the classical examples. We study here the basic properties of the sequence spaces ℓ_p $(0 < p < 1)$ and the function spaces L_p for $0 \leq p < 1$. Part of the aim of the chapter is to familarize the reader with some of the techniques that can be used in studying F-spaces. We also show as motivation for Chapter 4 that the Hahn-Banach theorem fails in each space (i.e. there is a continuous linear functional defined on a linear subspace which cannot be extended continuously to the whole space). For the spaces L_p, this amounts to showing that they have trivial dual spaces, i.e. every continuous linear functional is zero. However the sequence spaces ℓ_p have rich dual spaces, which we calculate; therefore to demonstrate the failure of the Hahn-Banach theorem requires rather more effort. In particular, we show that ℓ_p has a quotient with trivial dual and this implies the failure of the Hahn-Banach theorem.

Chapter 3: The Hardy Spaces H^p. The Hardy spaces H^p for $0 < p < 1$ provide another rich store of examples and have played an influential role in determining the general direction taken by F-space theory. We present here their more fundamental properties. In particular we calculate their "Banach envelopes" and demonstrate again the failure of the Hahn-Banach theorem. The spaces H^p have, like ℓ_p, rich dual spaces but again we show that they have quotients with trivial dual. As part of the chapter we present an elegant result of Aleksandrov that $H^p + H^p = L_p$.

Chapter 4: The Hahn-Banach Extension Property. Motivated by the results of Chapters 2 and 3, we show that an F-space in which the Hahn-Banach theorem holds is locally convex. Curiously this result is false without the assumption of metrizability and it is unknown whether completeness is necessary. A separable non-locally convex F-space which has a dual rich enough to separate points always has a quotient space with trivial dual.

Chapter 5: The three-space problem. A closed subspace N of an F-space X is said to have the Hahn-Banach Extension Property (HBEP) if every continuous linear functional on N can be extended to a continuous linear functional on X. The main result of Chapter 4 could be restated as saying that every non-locally convex F-space has a closed subspace which fails HBEP.

In Chapter 5 we study conditions on the quotient space X/N for a closed subspace N which imply that N has HBEP; a space Y is called a K-space if $X/N \approx Y$ implies N has HBEP. The main results of the chapter are that the spaces ℓ_p, L_p $(0 < p < \infty, \neq 1)$ are K-spaces. However an example due to Ribe is constructed to show that ℓ_1 is not a K-space; precisely, the Ribe space is a non-locally convex quasi-Banach space X which has a subspace N of dimension one so that $X/N \approx \ell_1$. The Ribe space is also an example of a separable F-space with no (non-trivial) quotient space with trivial dual.

Chapter 6: Lifting theorems. This short chapter reformulates some of the ideas of the previous chapter as lifting theorems for operators. We prove some general lifting theorems for L_p $(0 < p < 1)$ and deduce that L_p is a K-space.

Chapter 7: Transitivity and small operators. In Chapter 7 we broaden our interests from linear functionals to general compact operators. The general theme of the chapter is that it is quite difficult to find compact operators on spaces which do not already admit linear functionals. To make this notion precise we introduce the important idea of transitivity. X is transitive if for every $x_1, x_2 \in X$ with $x_1 \neq 0$ there is an endomorphism T of X so that $Tx_1 = x_2$. We show that on a transitive space with trivial dual there are no non-zero strictly singular (and hence no compact) endomorphisms; the reason is essentially that the Fredholm theorem can be extended to this setting (with some modifications). The spaces L_p $(0 < p < 1)$ are transitive, but for these spaces we prove stronger results. Any non-zero operator $T: L_p \to X$ (for any range space) is an isomorphism on an infinite-dimensional Hilbertian subspace of L_p.

To put these results in perspective we also construct a space with trivial dual which does have non-zero compact endomorphisms. This space is of course not transitive. A more spectacular non-transitive space is the rigid space constructed at the end of the chapter: this is a quasi-Banach space whose algebra of endomorphisms consists simply of multiples of the identity operator.

Chapter 8: Operators on L_p, $0 < p < 1$. The main theorems of this chapter give an explicit representation for all the endomorphisms of L_p for $0 < p < 1$. Using these theorems we can extend the results of Chapter 7. We see that every non-zero endomorphism of L_p $(0 < p < 1)$ actually preserves a copy of the whole space. We also sketch a proof that every non-zero operator $T : L_p \to L_0$ for $0 < p < 1$ preserves a copy of any L_q for $p < q \leq 2$.

Chapter 9: A compact convex set with no extreme points. The final chapter is devoted to a distinct but not unrelated question: can one construct in a non-locally convex F-space a compact convex set which fails to have extreme points? The answer is yes and we construct an example in L_0. The crucial notion is that of a needle-point, which we first met in disguised form in Chapter 5 (the Ribe space).

At this point it seems appropriate to make some apologies. Some readers will feel that certain topics should have been (or should not have been!) covered - obviously the selection of the material in a volume of limited length has to be somewhat personal. To the authors, it seems that one obvious topic which might have been included is the Maurey-Nikishin factorization theory for operators into L_0 (briefly mentioned in Chapter 8). However, we wished to keep as closely as possible to certain predetermined themes.

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N. J. Kalton, N. T. Peck and James W. Roberts
Frontmatter
More information

Although these notes have three distinct authors, we have tried to be as consistent as possible. There are inconsistencies in style and notation, which we hope will not unduly distract the reader. We all wish to apologize on behalf of each other for any illiteracy or obscurity!

Finally, it is a pleasure to thank: Allen Butler, Raouf Eldeeb, Larry Riddle, Jon Snader, David Trautman, who participated in courses or seminars on this material and contributed useful ideas; Patricia Coombs, for her expert typing of the manuscript; David Tranah, of the Cambridge University Press, for his advice and assistance; and the National Science Foundation, for its support.

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