CHAPTER I
PRELIMINARIES

1. Topological vector spaces

A topological vector space X over a field $\mathbb{K}$ (either the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$) is a vector space such that the operations of addition $(x,y) \to x + y$ $(X \times X \to X)$ and scalar multiplication $(\lambda,x) \to \lambda x$ $(\mathbb{K} \times X \to X)$ are jointly continuous. We shall assume that the reader is familiar with the basic properties of topological vector spaces, as expounded for example in Kothe [1969] or Rudin [1973] Chapter 3. In the introduction we present selected basic material which will be of special interest to us, omitting most of the routine proofs.

We recall that a vector topology τ on X can be described in terms of a base of neighborhoods $\mathcal{U}$ at the origin 0. A base $\mathcal{U}$ has the properties

(1) Given $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ with $V + V = \{v_1 + v_2 : v_1, v_2 \in V\} \subset U$.

(2) Given $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$, with $\alpha V \subset U$ for all $\alpha \in \mathbb{K}$ with $|\alpha| \leq 1$.

(3) Given $U \in \mathcal{U}$ and $x \in X$ there exists $n \in \mathbb{N}$ with $x \in nU$ (i.e. U absorbs $\{x\}$).

Conversely given a collection of sets $\mathcal{U}$ obeying (1)-(3), there is a vector topology τ on X with $\mathcal{U}$ as a neighborhood-base for the origin.

It is always possible to choose a base of neighborhoods of sets which are closed and balanced, i.e. $\lambda U \subset U$ for $|\lambda| \leq 1$.

(X,τ) is Hausdorff if and only if $\cap\, \mathcal{U} = \{0\}$. In general $\cap\, \mathcal{U}$ is a linear subspace of X. (X,τ) is locally convex if it has a base of neighborhoods $\mathcal{U}$ consisting of convex sets ; a set C is convex if $\lambda C + (1-\lambda)C \subset C$ for

$0 \leq \lambda \leq 1$. It is then possible to choose a base of *absolutely convex sets* U (i.e. such that $\lambda U + \mu U \subset U$ for $|\lambda| + |\mu| \leq 1$).

If M is a linear subspace of X then the quotient space X/M may be topologized by the quotient topology whose neighborhood base consists of all sets $(q(U) : U \in \mathcal{U})$ where $q : X \to X/M$ is the quotient map and $\mathcal{U}$ is a neighborhood base in X. X/M will be Hausdorff provided M is closed.

2. *Metric linear spaces*

We start with a rather non-standard definition which appears to be useful. Let X be a vector space. Then a Δ-*norm* is a map $x \to ||x||$ $(X \to \mathbb{R})$ so that

(1.1) $\|x\| > 0$ $\quad x \neq 0$

(1.2) $\|\alpha x\| \leq \|x\|$ $\quad |\alpha| \leq 1,\ x \in X$

(1.3) $\lim_{\alpha \to 0} \|\alpha x\| = 0$ $\quad x \in X$

(1.4) $\|x+y\| \leq C \max(\|x\|, \|y\|)$ $\quad x, y \in X$

where C is some constant independent of x, y. Note that $C > 1$.

If $\|\cdot\|$ is a Δ-norm on X then it induces on X a vector topology τ which is metrizable. A base of neighborhoods at the origin is given by sets of the form $U_n = \{x \in X : \|x\| < 1/n\}$. A sequence $x_n \in X$ converges to $x \in X$ if and only if $\|x - x_n\| \to 0$.

Conversely suppose τ is a topology with a countable base of neighborhoods (U_n) such that $\cap U_n = \{0\}$, each U_n is balanced and $U_{n+1} + U_{n+1} \subset U_n$ for every n. Then we can define a Δ-norm on X by

$$\|x\| = \sup(2^{-n} : x \notin U_n)$$

and the Δ-norm induces the original topology; here $C = 2$.

A Δ-norm is called an F-norm if it satisfies

$$(1.5) \qquad \|x+y\| \leq \|x\| + \|y\| \qquad x,y \in X.$$

If $\|\cdot\|$ is any F-norm then $d(x,y) = \|x-y\|$ is a (translation-)invariant metric on X. We first prove a metrization theorem which allows us to replace any Δ-norm with an F-norm.

LEMMA 1.1. Let $\|\cdot\|$ be any Δ-norm on X. Choose p so that $2^{1/p} = C$. Then for any $x_1,\ldots,x_n \in X$ we have

$$(1.6) \qquad \|x_1 + \ldots + x_n\| \leq 4^{1/p}(\|x_1\|^p + \ldots + \|x_n\|^p)^{1/p}.$$

Proof. By induction on (1.4) we can obtain

$$(1.7) \qquad \|x_1 + \ldots + x_n\| \leq \max_{1\leq k\leq n} C^k\|x_k\|.$$

for $x_1,\ldots x_n \in X$.

Let us define a function $H : X \to \mathbb{R}$ by

$$H(x) = 2^{n/p} \quad \text{if} \quad 2^{n-1/p} < \|x\| \leq 2^{n/p} \qquad n \in \mathbb{Z}$$

$$H(0) = 0.$$

Then

$$\|x\| \leq H(x) \leq 2^{1/p}\|x\|.$$

We shall show by induction that

$$(1.8) \qquad \|x_1 + \ldots + x_n\| \leq 2^{1/p}(H(x_1)^p + \ldots + H(x_n)^p)^{1/p}$$

and then (1.6) is immediate.

Of course (1.8) holds if $n = 1$. Suppose that it holds for $n = m$ and that $x_1, \ldots, x_{m+1} \in X$. We may suppose that $\|x_1\| \geqslant \|x_2\| \geqslant \ldots \geqslant \|x_{m+1}\|$. We consider two cases.

First suppose that the set of values $\{H(x_i) : 1 \leqslant i \leqslant m+1\}$ is distinct: then $H(x_i) \leqslant 2^{1-i/p} H(x_1)$ and so

$$\begin{aligned} C^i \|x_i\| &\leqslant C^i H(x_i) \\ &\leqslant 2^{1/p} H(x_1) \\ &\leqslant 2^{1/p} (H(x_1)^p + \ldots + H(x_n)^p)^{1/p} \end{aligned}$$

and (1.8) follows from (1.7).

Alternatively we have $H(x_j) = H(x_{j+1})$ for some j, $1 \leqslant j \leqslant m$. Hence, for some $r \in Z$,

$$2^{(r-1)/p} < \|x_{j+1}\| \leqslant \|x_j\| \leqslant 2^{r/p}$$

and so

$$\|x_j + x_{j+1}\| \leqslant 2^{(r+1)/p}.$$

Thus $H(x_j + x_{j+1})^p \leqslant H(x_j)^p + H(x_{j+1})^p$. Applying the inductive hypothesis,

$$\|x_1 + \ldots + x_{m+1}\|^p \leqslant 2\Big(\sum_{i \neq j, j+1} H(x_i)^p + H(x_j + x_{j+1})^p \Big)$$

and hence (1.8) follows for $n = m + 1$. □

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THEOREM 1.2. Let $\|\cdot\|$ be a Δ-norm on X. Then if p is chosen so that $2^{1/p} = C$, the formula

$$(1.9) \qquad |||x||| = \inf\left(\sum_{i=1}^{n} \|x_i\|^p : \sum x_i = x \right)$$

defines an F-norm on X giving the same topology.

<u>Proof</u>. Simply note that $(1/4)\|x\|^p \leq |||x||| \leq \|x\|^p$. □

COROLLARY. Let X be a Hausdorff topological vector space with a countable base of neighborhoods of 0. Then X is metrizable and the topology may be given by an invariant metric.

A metrizable topological vector space (or metric linear space) is called an F-<u>space</u> if it is complete for an invariant metric (and hence for every invariant metric.) An elegant result of Klee [1952] allows one to replace an invariant metric by <u>any</u> metric giving the topology. Thus a metric linear space which is complete for any metric will also be complete in any invariant metric.

Every metrizable topological vector space X can be embedded as a dense linear subspace of an F-space $\tilde{X}$. The construction of $\tilde{X}$ is simply to form the normal metric space completion of X with respect to an invariant metric and extend the vector space operations in the obvious way. The space $\tilde{X}$ obtained in this way is unique; it does not depend on the particular choice of invariant metric.

If N is a closed subspace of a metrizable topological vector space then X/N is also metrizable. Further if X is an F-space, X/N is an F-space. Again we shall not prove these statements which are left as an exercise.

3. Locally bounded spaces

A subset B of a (Hausdorff) topological vector space X is bounded if given any zero-neighborhood U we have $B \subset nU$ for some integer n. Note that if $B_1,\dots,B_m$ are bounded sets then so is $B_1 + \dots + B_m$.

Suppose X has a neighborhood of the origin B which is bounded. Then the sets $(1/n\, B : n \in \mathbb{N})$ form a neighborhood-base at the origin. Hence X is metrizable. Also $B+B$ is bounded and so for some $\rho \geq 1$, $B + B \subset \rho B$. Define a Δ-norm on X by using the Minkowski-function of B:

$$(1.10) \qquad \|x\| = \inf\{\lambda : \lambda^{-1}x \in B\}.$$

To see this is a Δ-norm note that

$$(1.11) \qquad \|x+y\| \leq \rho \max(\|x\|,\|y\|).$$

In addition $\|\cdot\|$ satisfies

$$(1.12) \qquad \|\alpha x\| = |\alpha|\,\|x\| \qquad \alpha \in \mathbb{K},\ x \in X.$$

A Δ-norm which satisfies (1.12) is called a quasi-norm. Conversely the topology associated with any quasi-norm is locally bounded. A complete locally bounded space is called a quasi-Banach space.

For $0 < p \leq 1$ we say that a subset C of a vector space X is p-convex if whenever $x_1,x_2 \in C$ and $a_1,a_2 \in \mathbb{R}$ with $a_1 \geq 0$, $a_2 \geq 0$, $a_1^p + a_2^p = 1$, then $a_1x_1 + a_2x_2 \in C$. C is absolutely p-convex if $a_1x_1 + a_2x_2 \in C$ whenever $x_1,x_2 \in C$, $a_1,a_2 \in \mathbb{K}$ and $|a_1|^p + |a_2|^p \leq 1$.

Suppose now that a locally bounded space X has a bounded absolutely p-convex neighborhood B of the origin.

Then the quasi-norm given by (1.10) satisfies

(1.13) $$\|x+y\|^p \leqslant \|x\|^p + \|y\|^p \qquad x,y \in X.$$

Conversely, if a quasi-norm $\|\cdot\|$ satisfies (1.13) then its unit ball $U = \{x : \|x\| \leqslant 1\}$ is absolutely p-convex. We say that the quasi-norm is then p-_subadditive_ and the locally bounded space X is _locally_ p-_convex_ or merely p-_convex_. It is common to abbreviate "p-convex quasi-Banach space" to "p-_Banach space_," and it will be assumed, when this term is used, that the associated quasi-norm satisfies (1.13).

If we specialize Theorem 1.2 to this situation we have a classical result due to Aoki [1942] and Rolewicz [1957].

THEOREM 1.3. (Aoki-Rolewicz theorem). If X is a locally bounded space then X is p-convex for some $p > 0$. Precisely, if B is a bounded neighborhood of the origin with $B + B \subset \rho B$ then X is p-convex where $2^{1/p} = \rho$.

Proof. It follows from Theorem 1.2 that if

$$\|x\|_1 = \inf\left(\left(\sum_{i=1}^{n} \|x_i\|^p\right)^{1/p} : \sum x_i = x\right)$$

then $\|x\|_1$ is a p-subadditive quasi-norm equivalent to the given quasi-norm. □

REMARKS. The constant ρ in Theorem 4.3 is such that

(1.14) $$\|x+y\| \leqslant \rho \max(\|x\|,\|y\|) \qquad x,y \in X.$$

However it is more common to associate to the quasi-norm a constant k, the _modulus of concavity_ of the quasi-norm, so that

(1.15) $$\|x+y\| \leqslant k(\|x\|+\|y\|) \qquad x,y \in X.$$

Of course, (1.14) implies (1.15) with $k = \rho$ while (1.15) implies (1.14) with $\rho = 2k$.

However Theorem 1.3 yields the fact that if (1.14) holds then X may be equivalently quasi-normed by a p-subadditive quasi-norm $\| \ \|_1$ and then

$$\|x+y\|_1 \leq (\|x\|_1^p + \|y\|_1^p)^{1/p}$$

which implies by elementary calculus

$$\|x+y\|_1 \leq 2^{1/p-1}(\|x\|_1 + \|y\|_1)$$

$$= 1/2\ \rho(\|x\|_1 + \|y\|_1).$$

Thus by renorming we can take $k = 1/2\ \rho$. Note also that from Theorem 1.3 if (1.15) holds then X is p-convex where $2^{1/p} = 2k$ or $k = 2^{1/p-1}$.

4. Linear operators and the closed graph theorem

If X and Y are topological vector spaces (over the same base field $\mathbb{K}$), then a linear map $T : X \to Y$ will be called an operator when it is continuous. This is equivalent to the statement that for every zero-neighborhood U in Y, $T^{-1}(U)$ is a zero-neighborhood in X. When X and Y are metrizable and their topologies are given by Δ-norms (both denoted by $\|\cdot\|$) then T is continuous if given $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ so that $\|x\| < \delta$ implies $\|Tx\| < \epsilon$. If X and Y are both locally bounded and have their topologies given by quasi-norms we need only have

$$\|T\| = \sup_{\|x\|\leq 1} \|Tx\| < \infty.$$

We denote by $\mathcal{L}(X,Y)$ the vector space of all linear operators $T : X \to Y$. If $Y = \mathbb{K}$, the base field, we call an operator a linear functional and write $\mathcal{L}(X,\mathbb{K}) = X^*$. If $Y = X$

then an operator is an endomorphism and $L(X,X)$ is abbreviated to $L(X)$.

In this section we shall prove the familiar Open Mapping and Closed Graph Theorems for F-spaces. Although these are to be found in any basic functional analysis book, we shall need certain strengthened forms which are perhaps less well known.

THEOREM 1.4. (The Open Mapping Theorem). Let X be an F-space and let Y be a Hausdorff topological vector space. Let $T : X \to Y$ be a linear operator such that for every zero-neighborhood U in X, $\overline{T(U)}$ is a neighborhood of zero in Y. Then

(1) T is an open mapping i.e. $T(U)$ is a neighborhood of 0 for every neighborhood U of zero in X,
(2) Y is an F-space.

<u>Proof</u>. (Essentially as in Rudin [1973], p. 47). Suppose X is F-normed by $\|\cdot\|$. Let

$$V(\epsilon) = \{x \in X : \|x\| < \epsilon\}.$$

We show $T[V(\epsilon)] \supset \overline{T[V(1/2\ \epsilon)]}$. Indeed suppose $y = y_1 \in \overline{T[V(1/2\ \epsilon)]}$. By induction we pick $y_n \in \overline{T[V(2^{-n}\epsilon)]}$, and $x_n \in V(2^{-n}\epsilon)$. Suppose y_n has been chosen. Then

$$(y_n + \overline{T[V(2^{-n-1}\epsilon])}) \cap T[V(2^{-n}\epsilon] \neq \varphi.$$

Pick $x_n \in V(2^{-n}\epsilon)$ so that

$$y_n - Tx_n \in \overline{T[\cdot\ 2^{-n-1}\epsilon)]}$$

and let $y_{n+1} = y_n - Tx_n$.

Now $\|x_n\| < 2^{-n}\epsilon$ and so

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$$x = \sum_{n=1}^{\infty} x_n$$

exists in X and

$$\|x\| < \epsilon.$$

T is continuous, so that

$$\begin{aligned} Tx &= \sum_{n=1}^{\infty} Tx_n \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} Tx_n \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} (y_n - y_{n+1}) \\ &= \lim_{N\to\infty} (y_1 - y_{N+1}). \end{aligned}$$

However $y_{N+1} \in \overline{T[V(2^{-N-1}\epsilon)]}$ and so as T is continuous $y_{N+1} \to 0$. Thus $Tx = y_1 = y \in T[V(\epsilon)]$. This immediately establishes (1). Now it follows that $T(V(2^{-n}))$ is a base of neighborhoods in Y so that Y is isomorphic to the quotient $X/\ker T$ (where $\ker T = \{x : Tx = 0\}$). Hence Y is an F-space. □

COROLLARY 1.5. Let X and Y be F-spaces and let $T : X \to Y$ be a surjective linear operator. Then T is an open mapping.

<u>Proof</u>. If U is a neighborhood of 0 in X, then choose another balanced neighborhood V with $V + V \subset U$. Then $\cup\, nT(V) = Y$ and so by the Baire Category Theorem, for some k $\operatorname{int} \overline{kT(V)}$ is non-empty. Thus $\operatorname{int} \overline{T(V)}$ is non-empty and so 0 is in the interior of $\overline{T(V)} - \overline{T(V)} \subset \overline{T(U)}$. Now apply Theorem 1.4. □

THEOREM 1.6. (The Closed Graph Theorem). Let X be a Hausdorff topological vector space and let Y be an F-space. Let