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Homogeneous Structures on Riemannian Manifolds

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To Magda and Nuccia

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PREFACE

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INTRODUCTION

As is well-known, E. Cartan proved that a connected, complete and simply connected Riemannian manifold is a symmetric space if and only if the curvature is constant under parallel translation. In 1958 Ambrose and Singer [1] extended this theory and gave a characterization of *homogeneous Riemannian manifolds* by a local condition which is to be satisfied at all points. More specifically, they proved that a connected, complete and simply connected Riemannian manifold (M,g) is homogeneous, i.e. there exists a transitive and effective group G of isometries of M, if and only if there exists a tensor field T of type (1,2) such that

$$\text{(AS)}\quad \begin{cases} \text{(i)} & g(T_XY,Z) + g(Y,T_XZ) = 0\ , \\ \text{(ii)} & (\nabla_X R)_{YZ} = [T_X,R_{YZ}] - R_{T_XYZ} - R_{YT_XZ}\ , \\ \text{(iii)} & (\nabla_X T)_Y = [T_X,T_Y] - T_{T_XY}\ , \end{cases}$$

for $X,Y,Z \in \mathfrak{X}(M)$. Here ∇ denotes the Levi Civita connection and R is the Riemann curvature tensor of M.

These conditions are also used to study weakly locally homogeneous, infinitesimally homogeneous and curvature homogeneous manifolds [1],[44].

Although there are shorter proofs than that given in [1] (see for example [28]) the treatment of Ambrose and Singer has the advantage of being *constructive*. Indeed, the authors give an explicit construction of the tensor T when the group G is given and conversely, they determine the group G when a particular T is given. In chapter 1 of these notes we give a short proof of the theorem of Ambrose and Singer but we also give a full version of the constructive proof because on the one hand a lot of geometers are not familiar with it and on the other hand some important results are needed in the rest of these notes.

The proof of Ambrose and Singer sets up a natural correspondence between groups G of which M is a homogeneous Riemannian manifold and the T's which satisfy the conditions (AS). In that paper the authors make the following remark : "This suggests the possibility of classifying the groups G of which M is a Riemannian homogeneous space thru the T's. It also suggests the possibility of classifying Riemannian homogeneous manifolds by properties of the T's." They give two examples concerning this last suggestion. We will come back on these examples later on.

Our interest in this theorem and these remarks arose during our research on *harmonic spaces* and *spaces with volume-preserving geodesic symmetries* (see [51]). First of all we wanted to have a method to be able to decide whether a harmonic manifold is a homogeneous space or not. Secondly we tried to find manifolds with volume-preserving geodesic symmetries which are not naturally reductive. To decide whether a homogeneous manifold is naturally reductive is far from easy in many cases and therefore we were looking for a characterization using the tensors T instead of using all the groups of isometries of the Riemannian manifold. At the same time we wanted to have a method which was closely related to the curvature and the geometric or Riemannian properties of the manifold.

The first suggestion made by Ambrose and Singer gives rise to a difficult problem and this for several reasons. Let (M,g) be a homogeneous Riemannian space with a given group G of isometries. Following the method of Ambrose and Singer, this determines a tensor field T. Now using this T one can determine conversely the group of isometries and this is in general a group G' which is not isomorphic to G. The Euclidean plane $\mathbb{R}^2$ provides a simple example. Let G denote the group of all isometries of $\mathbb{R}^2$. Then one obtains T = 0. Further, the construction of G' starting from T = 0 gives for G' only the group of translations of $\mathbb{R}^2$. Hence an important problem will be to understand for which spaces we have G = G'. Only for this case the solutions of the equations (AS) will give a parametrization of the transitive and effective groups G. This means also that we have to understand which groups can be obtained from the solutions of the equations (AS) and how we can characterize these groups. We refer to chapter 1 and chapter 2 for more detailed information about this problem.

Concerning the second suggestion they consider the example of homogeneous Riemannian manifolds such that for all $X,Y \in \mathfrak{X}(M)$ we have

$$T_X Y + T_Y X = 0.$$

Moreover, they prove that for these spaces the geodesics of the Riemannian connection are orbits of one-parameter subgroups of G. In chapter 9 we discuss a remarkable example of A. Kaplan which shows that the converse property does not hold. In fact we will prove that the condition for T characterizes naturally reductive homogeneous spaces. This implies that Theorem 5.4 in [1] has to be modified (see chapter 6).

This condition on the T is an *algebraic* condition which is invariant under the action of the orthogonal group. For this reason we study the decomposition of the space of tensors T which satisfy the condition ((AS)(i)) into irreducible factors under the action of the orthogonal group. In this way we obtain a set of algebraic conditions for the tensor T. These conditions are invariant under the action of the orthogonal group and they provide a kind of *classification* for the homogeneous Riemannian spaces into eight different classes. This method is similar to that used in [14] for the study of Einstein-like manifolds and in [15] to give a classification of almost Hermitian manifolds into sixteen classes. See [49] for a similar treatment of the space of curvature tensors on an almost Hermitian manifold but under the action of the unitary group. See also [6] for the orthogonal group.

In chapter 2 we first define and treat *homogeneous Riemannian structures* from a general viewpoint. Such a structure is given on a Riemannian manifold by a solution of the equations (AS). Note that a solution of the equations (AS), in general, is not uniquely determined. For example, let T be a homogeneous Riemannian structure on (M,g) and φ an isometry of M. Then the tensor T' given by

$$T'_X Y = \varphi_* T_{\varphi_*^{-1} X} \varphi_*^{-1} Y ,$$

$X,Y \in \mathfrak{X}(M)$, determines also a homogeneous Riemannian structure and, in general, T' is different from T. This leads to the definition of *isomorphic homogeneous structures*. But it also can happen that on the same manifold (M,g) there exist two *nonisomorphic* homogeneous structures T and T'. The point is that T and T' give rise to two nonisomorphic transitive groups of isometries or to the same group but with different reductive decompositions of the Lie algebra of the group. We refer to chapters 7 and 8 for detailed examples of these possibilities.

Chapter 3 contains the *algebraic* part. Here we give the decomposition mentioned earlier, we determine the quadratic invariants and we write down the projections of T on the irreducible factors.

In chapter 4 we concentrate on the homogeneous Riemannian structures on *two-dimensional manifolds* and give the complete classification. More specifically, we show that the Poincaré half-plane is the only connected, complete and simply connected surface which has a homogeneous Riemannian structure T *different from zero*. Moreover, up to isomorphism, this nonvanishing structure is unique.

Next, in chapter 5, we study the class $\mathcal{T}_1$ of homogeneous structures with defining condition

$$T_X Y = g(X,Y)\xi - g(Y,\xi)X ,$$

where ξ is a given vector field on (M,g) and $X,Y \in \mathfrak{X}(M)$. This is the immediate analogue of the case for surfaces since for algebraic reasons all the homogeneous structures on surfaces are of this type. We prove that the *hyperbolic space* is the only space (connected, complete, simply connected) having such a structure $T \neq 0$.

The main result of chapter 6 is the characterization of *naturally reductive homogeneous* spaces by the condition "$T_X X = 0$ for all $X \in \mathfrak{X}(M)$". These are by far the simplest kind of homogeneous spaces. There are many examples known. For example, all the homogeneous Riemannian spaces whose isotropy representation is irreducible (in particular, the irreducible symmetric spaces) belong to this class. See the classification given by J. Wolf [57]. See also [12] and chapters 6,7 and 8. Further we give a complete list of all the *three-dimensional* connected, complete and simply connected manifolds which admit such a nonvanishing structure. The main point is that this is done here with the help of the curvature tensor R of the manifold and the tensor T, concentrating in this way on the Riemannian viewpoint.

In chapter 7 and chapter 8 we treat in detail some important examples. Chapter 7 is completely devoted to the study of the *Heisenberg group*. In chapter 8 we consider all the *Lie groups of dimension three*, the 3-*symmetric spaces*, the *four-dimensional hyperbolic space* and some other *four-dimensional Lie groups*. On the one hand these examples illustrate several general statements and theorems and on the other hand they provide examples for the eight classes and their respective inclusions.

Motivated by our research on manifolds with volume-preserving geodesic symmetry (see [51]) A. Kaplan discovered a nice six-dimensional example with this property but which is not naturally reductive. The same example shows that there are manifolds such that all geodesics are orbits of one-parameter subgroups of isometries which are not naturally reductive. This example is a *group of type* H or a *generalized Heisenberg group*. In chapter 9 we give a brief survey on these groups and prove several of Kaplan's results. We do this in a different way using the methods of these notes. Finally we show that the six-dimensional example has some new additional properties for the geodesic symmetries which are much stronger than the volume-preserving property. These properties are also valid for naturally reductive homogeneous spaces and hence, the example shows again that the given properties are not characteristic for naturally reductive spaces.

As is now well-known, the decomposition of the space of curvature tensors on a Riemannian manifold has three irreducible invariant components under the action of the orthogonal group. When one considers four-dimensional manifolds and the special orthogonal group, then one of the components splits further into two irreducible spaces of the same dimension. This gives rise to the notion of self-dual and anti-self-dual curvature tensors. In chapter 10 we consider the space of tensors T as before but now under the action of the special orthogonal group and for dimension four. We show that also in this case one of the irreducible factors for O(4) splits into two irreducible spaces of the same dimension for SO(4). This leads to the definition of *self-dual* and *anti-self-dual homogeneous structures*. We provide examples of such structures.

There are several problems which need further research. We mention the following ones here. In the first place it is shown in chapter 6 how the manifolds with a structure of type $\mathcal{T}_3$, i.e. a naturally reductive structure, can be characterized by means of properties of the geodesics or geometrical notions related to the geodesics. It would be interesting to have similar characterizations for all the other classes. Secondly, further research for examples is needed to consider the inclusion relations for the sixteen classes in chapter 10 and to lend more substance to the introduction of self-dual and anti-self-dual homogeneous structures.

The bibliography has been kept to a minimum and consists only of papers and books referred to in these notes. Further information, and a more complete list of papers on homogeneous manifolds, can be found in the books and papers cited.