1. THE THEOREM OF AMBROSE AND SINGER

The main purpose of this section is to give a proof of the theorem of Ambrose and Singer and to concentrate on several facts we need later on. There are three sections. In section A we collect some preliminary material used in the two other sections and in the rest of these notes. In section B we give a more direct proof of the theorem, inspired by the method used in [27],[28] in a more general context. (See also [37],[38].) Finally in section C we consider the proof given by Ambrose and Singer with slight modifications. More specifically, we give the explicit construction of the transitive group G of isometries and its algebra $\mathfrak{g}$ when the tensor field T is given. This is a fundamental construction for the development of the theory of homogeneous structures. At many places we refer to [26] as a standard reference for a lot of notions, theorems and additional material.

A. PRELIMINARIES

Let (M,g) be an n-dimensional Riemannian manifold of class C^∞. Let ∇ denote the Levi Civita connection of (M,g) and R the corresponding Riemann curvature tensor given by

$$R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y] , \qquad X,Y \in \mathfrak{X}(M) , \tag{1.1}$$

where $\mathfrak{X}(M)$ is the Lie algebra of C^∞ vector fields on M.

Let φ be an *isometry* of (M,g). Then φ is also an *affine transformation* of ∇. This means :

$$\varphi_*(\nabla_X Y) = \nabla_{\varphi_* X}\varphi_* Y \tag{1.2}$$

for $X,Y \in \mathfrak{X}(M)$. Here φ_* denotes the differential of φ.

A vector field ξ on M is called an *infinitesimal isometry* or a *Killing vector field* if the local one-parameter group of local transformations generated by ξ in a neighbourhood of each point of M consists of local isometries. This is equivalent to the following condition :

$$(\mathcal{L}_\xi g)(X,Y) = \xi g(X,Y) - g([\xi,X],Y) - g(X,[\xi,Y]) = 0 \ , \qquad (1.3)$$

$X,Y \in \mathfrak{X}(M)$. $\mathcal{L}_\xi$ denotes the *Lie derivative* with respect to ξ. Now put

$$A_X Y = -\nabla_Y X = \mathcal{L}_X Y - \nabla_X Y. \qquad (1.4)$$

Then (1.3) becomes

$$g(A_\xi X,Y) + g(X,A_\xi Y) = 0. \qquad (1.5)$$

Hence ξ is a Killing vector field if and only if A_ξ is *skew-symmetric*.

It is well-known (see for example [26, vol. I, p. 239]) that the *group* $\mathfrak{I}(M)$ *of all isometries* of a Riemannian manifold is a Lie group of transformations of M. The Lie algebra i(M) of $\mathfrak{I}(M)$ is isomorphic to the Lie algebra of *complete Killing vector fields*.

Next we recall

PROPOSITION 1.1. *Let* (M,g) *be a connected Riemannian manifold and* φ,ψ *two isometries of* M. *Suppose there exists a point* p *of* M *such that*

$$\varphi(p) = \psi(p) \ , \quad \varphi_*(p) = \psi_*(p).$$

Then $\varphi = \psi$.

For a proof see [19, p. 62]. It follows that an isometry of a connected Riemannian manifold (M,g) is completely determined by its value and its differential at a single point. The following proposition gives an infinitesimal version of this result.

PROPOSITION 1.2. *Let* ζ *and* ξ *be two Killing vector fields on a connected Riemannian manifold* M. *Suppose*

$$\zeta|_p = \xi|_p \quad , \quad A_\zeta|_p = A_\xi|_p$$

for some $p \in M$. *Then* $\zeta = \xi$.

We refer to [27] for a proof.

Among the affine transformations of (M,∇) the isometries of (M,g) are characterized as follows.

PROPOSITION 1.3. *Let* (M,g) *be a connected Riemannian manifold and let* φ *be an affine transformation with respect to the Levi Civita connection* ∇. *Suppose there exists a point* p *of* M *where* $\varphi_*(p)$ *is an isometry. Then* φ *is an isometry.*

Note that the fact that the *parallel transport* with respect to ∇ along a curve is an isometry is the important point in the proof (see [19, p. 201]). This is still true if ∇ is replaced by an arbitrary metric connection $\tilde{\nabla}$, i.e. $\tilde{\nabla}g = 0$. Hence Proposition 1.3 is also valid if one replaces ∇ by $\tilde{\nabla}$.

We will also need the following proposition. We recall briefly the proof of [1].

PROPOSITION 1.4. *Let* (M,g) *be a complete Riemannian manifold. If* X *is a vector field such that its norm* $\|X\|^2 = g(X,X)$ *is bounded, then* X *is a complete vector field.*

Proof. Let $\|X\| \leqslant k$ where k is a constant. Further, let $\varphi(t)$ be an integral curve of X defined for $a < t < b$. Now let $\{t_n\}$ be an infinite sequence converging to b. Then the sequence $\varphi(t_n)$ is a Cauchy sequence. Indeed, the distance between $\varphi(t_n)$ and $\varphi(t_m)$ is such that

$$d(\varphi(t_n),\varphi(t_m)) \leqslant \int_{t_n}^{t_m} \|\dot{\varphi}(t)\|\,dt \leqslant k(t_m - t_n).$$

Since (M,g) is complete, $\varphi(t_n)$ converges to a point p of M. It is clear that p is independent of the sequence $\{t_n\}$ chosen before.

Next choose the integral curve $\psi(s)$ of X through p and take

$\psi(0) = p$. This integral curve is defined on an interval $(-\varepsilon,\varepsilon)$. Now put

$$\tilde{\varphi}(t) = \begin{cases} \varphi(t) & \text{for } a < t < b , \\ \psi(t-b) & \text{for } b - \varepsilon \leqslant t < b + \varepsilon. \end{cases}$$

Then $\tilde{\varphi}(t)$ is an integral curve of X which is defined for all t in the interval $(a,b+\varepsilon)$. This implies that each *maximal* integral curve of X is necessarily defined for all $t \in \mathbb{R}$. Hence X is complete.

Concerning *complete* Riemannian manifolds we now give a proof of the following fact :

PROPOSITION 1.5. *Let* (M,g) *be a complete Riemannian manifold. Then each metric connection* $\tilde{\nabla}$ *on* M *is complete.*

Proof. This result is classical in the case when the metric connection $\tilde{\nabla}$ is the Levi Civita connection of (M,g) (see [26, vol. I, p. 172] or [41, p. 102]).

Now let $\tilde{\nabla}$ be an arbitrary metric connection and let $\gamma(t)$ be a geodesic of $\tilde{\nabla}$ defined for $a < t < b$. Choose an infinite sequence $\{t_n\}$ which has b as limit. Since $\tilde{\nabla}$ is metric, $\|\dot{\gamma}(t)\| = k$ where k is constant. It is easy to see that $\{\gamma(t_n)\}$ is a Cauchy sequence and hence it converges to a point p of M.

Let $\mathcal{U}$ be the (relatively compact) domain of a system of normal coordinates $(x^1,\dots,x^n)$ centered at p. Then the functions $\gamma^i(t) = x^i(\gamma(t))$, $i = 1,\dots,n$, are defined for $c < t < b$ where $a < c < b$. Since $\|\dot{\gamma}(t)\| = k$ is constant, the functions $\dot{\gamma}^i(t)$ are bounded. Hence, the functions $\ddot{\gamma}^i(t)$ are also bounded since

$$\ddot{\gamma}^i + \sum_{j,k} \tilde{\Gamma}^i_{jk}\dot{\gamma}^j\dot{\gamma}^k = 0.$$

Here the $\tilde{\Gamma}^i_{jk}$ are the local components of the connection $\tilde{\nabla}$. It follows from the mean value theorem that the $\dot{\gamma}^i(t)$ are uniformly continuous and this implies that $\lim \dot{\gamma}^i(t)$ exists when $t \to b$. Put

$$\lim_{t\to b} \dot{\gamma}^i(t) = a^i \quad , \quad i = 1,\dots,n$$

and let $u \in T_pM$ be the vector

$$u = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x_i}\Big|_p .$$

The geodesic of $\tilde{\nabla}$ which is tangent to u at p is given by

$$\psi(s) = (a^1 s, \dots, a^n s) , \quad -\varepsilon < s < \varepsilon.$$

Hence

$$\tilde{\gamma}(t) = \begin{cases} \gamma(t) & \text{for } a < t < b , \\ \psi(t-b) & \text{for } b \leqslant t < b + \varepsilon \end{cases}$$

is a curve of class C^2. Since $\tilde{\gamma}(t)$ satisfies the system of equations for a geodesic, it is an extension of $\gamma(t)$. This implies that $\tilde{\nabla}$ is complete.

In what follows we begin by concentrating on the *principal fibre bundle* $\mathcal{O}(M)$ of *orthonormal frames* of (M,g) and on some of its properties. The structure group of $\mathcal{O}(M)$ is the orthogonal group $O(n)$, where n is the dimension of M. A point u of $\mathcal{O}(M)$ is a pair $(p;u_1,\dots,u_n)$ where $p \in M$ and $(u_1,\dots,u_n)$ is an orthonormal frame of T_pM. The projection $\pi : \mathcal{O}(M) \to M$ is determined by $\pi(u) = p$. Further, let $\mathcal{U}$ be an open neighbourhood of $p = \pi(u)$ and let $(E_1,\dots,E_n)$ be a local orthonormal frame field on $\mathcal{U}$ (a *local cross section* of $\mathcal{O}(M)$). Then, for all $v = (q;v_1,\dots,v_n)$ of $\pi^{-1}(\mathcal{U})$, we have

$$v_h = \sum_m a_h^m E_m|_q , \tag{1.6}$$

where $a = (a_h^m)$ is an element of $O(n)$. Hence the map $v \mapsto (\pi(v),a)$ is a diffeomorphism of $\pi^{-1}(\mathcal{U})$ onto $\mathcal{U} \times O(n)$. So we may identify $\pi^{-1}(\mathcal{U})$ with $\mathcal{U} \times O(n)$ and the tangent space of $\mathcal{O}(M)$ at $v \in \pi^{-1}(\mathcal{U})$ can be identified with the direct sum

$$T_{\pi(v)}M \oplus T_a O(n).$$

Hence a tangent vector $\bar{X}$ of $T_v\mathcal{O}(M)$ can be expressed as

$$\bar{X} = X + aA , \tag{1.7}$$

where $X = \pi_*(\bar{X})$ and $A \in \mathfrak{so}(n)$. Here $\mathfrak{so}(n)$ is the Lie algebra of $O(n)$, identified as usual with the tangent space of $O(n)$ at the identity.

Recall that $O(n)$ acts *freely* (without fixed points) on $\mathcal{O}(M)$ and *transitively* (on the right) on the fibres $\pi^{-1}(p)$. The action of $O(n)$ is given by

$$(p;u_1,\ldots,u_n)b = (p; \sum_m b_1^m u_m,\ldots, \sum_m b_n^m u_m) \tag{1.8}$$

where $b = (b_k^h) \in O(n)$. Identifying $\pi^{-1}(\mathcal{U})$ with $\mathcal{U} \times O(n)$, we can write

$$(p,a)b = (p,ab). \tag{1.9}$$

Next let

$$u(t) = (p(t);u_1(t),\ldots,u_n(t))$$

be a curve of $\mathcal{O}(M)$. One says that $u(t)$ is a *horizontal curve with respect to a metric connection* $\tilde{\nabla}$ if all the vector fields $u_h(t)$, $1 \leqslant h \leqslant n$, are *parallel* along the curve $p(t)$ of M. (See for example [26, vol. I].) Hence $u(t)$ is horizontal if and only if, locally, we have

$$\dot{a}_k^h(t) + \sum_m \tilde{\omega}_m^h(\dot{\gamma}(t))a_k^m(t) = 0. \tag{1.10}$$

The $\tilde{\omega}_j^i$ are the local forms of the connection $\tilde{\nabla}$, i.e.

$$\tilde{\omega}_j^i(X) = g(\tilde{\nabla}_X E_j, E_i) \tag{1.11}$$

where $(E_1,\ldots,E_n)$ is the local section of $\mathcal{O}(M)$ which gives the identification of $\pi^{-1}(\mathcal{U})$ with $\mathcal{U} \times O(n)$.

A vector $\bar{X}$ of $T_u\mathcal{O}(M)$ is said to be *horizontal* if it is a tangent vector of a horizontal curve. Hence it follows from (1.7) and

(1.10) that

$$\bar{X} = X + aA \tag{1.12}$$

is horizontal if and only if

$$X = \pi_*(\bar{X}) \;, \tag{1.13}$$

$$A = -a^{-1}\tilde{\omega}(X)a \tag{1.14}$$

where $\tilde{\omega}(X)$ is the matrix $(\tilde{\omega}^i_j(X)) \in \mathfrak{so}(n)$.

The horizontal vectors generate a subspace H_u of $T_u\mathcal{O}(M)$, called the *horizontal space*. The *vertical space* is the subspace of $T_u\mathcal{O}(M)$ which is tangent to the fibre through u. We recall (see [26, vol. I]) that the map $u \mapsto H_u$ determines an *infinitesimal connection* on $\mathcal{O}(M)$, *associated to* $\tilde{\nabla}$, i.e. we have

i) H_u depends differentiably on u;

ii) $H_u \oplus V_u = T_u\mathcal{O}(M)$;

iii) $(R_b)_* H_u = H_{ub}$,

where $(R_b)_*$ denotes the differential of the right translation with $b \in O(n)$.

Now let $\xi = (\xi^1,\dots,\xi^n)$ be an element of $\mathbb{R}^n$. Each u of $\mathcal{O}(M)$ defines an isomorphism between $\mathbb{R}^n$ and T_pM, $p = \pi(u)$, as follows :

$$u(\xi) = \xi^1 u_1 + \dots + \xi^n u_n .$$

The *standard horizontal vector field* corresponding to ξ and with respect to $\tilde{\nabla}$ is the vector field $B(\xi)$ such that $B(\xi)|_u$ is the unique horizontal vector with

$$\pi_*(B(\xi)|_u) = u(\xi).$$

Further, let $A \in \mathfrak{so}(n)$. A^* denotes the *fundamental vector field* corresponding to A, i.e. it is the vertical vector field generated by the one-parameter group of transformations of $\mathcal{O}(M)$ determined by $u \mapsto u(\exp tA)$.

Note that (see [26, vol. I, p. 51 and p. 119])

$$(R_a)_* A^* = (\mathrm{ad}(a^{-1})A)^* , \qquad a \in O(n) , \tag{1.15}$$

$$(R_a)_* B(\xi) = B(a^{-1}\xi) , \quad a \in O(n), \ \xi \in \mathbb{R}^n. \tag{1.16}$$

The standard horizontal vector fields generate the horizontal distribution and the fundamental vector fields generate the vertical distribution. Hence we can define a *Riemannian metric* $g_{\tilde{\nabla}}$ on $\mathcal{O}(M)$, depending on $\tilde{\nabla}$, by putting

$$g_{\tilde{\nabla}}(B(\xi),B(\eta)) = \langle \xi,\eta \rangle , \quad \xi,\eta \in \mathbb{R}^n , \tag{1.17}$$

$$g_{\tilde{\nabla}}(A_1^*,A_2^*) = -\mathrm{tr}(A_1A_2) , \quad A_1,A_2 \in \mathfrak{so}(n) , \tag{1.18}$$

$$g_{\tilde{\nabla}}(B(\xi),A^*) = 0 , \qquad \xi \in \mathbb{R}^n, \ A \in \mathfrak{so}(n). \tag{1.19}$$

$\langle\ ,\ \rangle$ denotes the inner product on $\mathbb{R}^n$.

It is clear that the projection $\pi : \mathcal{O}(M) \to M$ is a *Riemannian submersion*. Hence if $\tilde{d}$ denotes the distance function of $(\mathcal{O}(M),g_{\tilde{\nabla}})$ and d the distance function of (M,g) respectively, we have

$$d(\pi(u),\pi(v)) \leqslant \tilde{d}(u,v). \tag{1.20}$$

This remark is important for the following proposition.

PROPOSITION 1.6. *Let* (M,g) *be complete. Then* $(\mathcal{O}(M),g_{\tilde{\nabla}})$ *is also complete.*

Proof. To prove that $(\mathcal{O}(M),g_{\tilde{\nabla}})$ is complete, it is sufficient to show that the closure $\bar{A}$ of a bounded subset A of $\mathcal{O}(M)$ is compact (see [26, vol. I, p. 172]). It follows from (1.20) that if A is bounded, then $\pi(A)$ is also bounded. Hence $\overline{\pi(A)}$ is compact. Further, since the fibre of $\mathcal{O}(M)$ is compact, $\pi^{-1}(\overline{\pi(A)})$ is compact ([47, p. 13]) and closed. This implies that $\bar{A} \subset \pi^{-1}(\overline{\pi(A)})$ and hence, since $\bar{A}$ is closed, it is compact.

REMARK. Proposition 1.5 also follows from Proposition 1.4 and Proposition 1.6. Indeed, $\tilde{\nabla}$ is complete if and only if any standard horizontal vector field $B(\xi)$ with respect to $\tilde{\nabla}$ is complete (see [26, vol. I, p. 140]). But since

$$\|B(\xi)\| = \|\xi\|_{\mathbb{R}^n} ,$$

$\|B(\xi)\|$ is bounded. This implies the result.

The definition of $g_{\tilde{\nabla}}$ and (1.15), (1.16) imply

PROPOSITION 1.7. *All the right translations* R_a, $a \in O(n)$, *act as isometries on* $(\mathcal{O}(M), g_{\tilde{\nabla}})$.

Next, let $\tilde{\nabla}$ be the Levi Civita connection ∇ of (M,g). Then we have

PROPOSITION 1.8. *Each isometry* φ *of* (M,g) *induces an isometry* $\tilde{\varphi}$ *of* $(\mathcal{O}(M), g_{\nabla})$ *by*

$$\tilde{\varphi}(u) = \tilde{\varphi}(p;u_1,\ldots,u_n) = (\varphi(p);\varphi_*(u_1),\ldots,\varphi_*(u_n)). \tag{1.21}$$

Proof. First we note that

$$\tilde{\varphi}(ua) = \tilde{\varphi}(u)a , \qquad u \in \mathcal{O}(M), \ a \in O(n).$$

Hence if $A \in \mathfrak{so}(n)$, we have

$$\tilde{\varphi}_{*|u}(A^*_{|u}) = A^*_{|\tilde{\varphi}(u)}. \tag{1.22}$$

Since φ is an isometry, it preserves the parallelism. So if X is a horizontal vector field of $\mathcal{O}(M)$, $\tilde{\varphi}_*(X)$ is also horizontal.

Next we have

$$\pi \circ \tilde{\varphi} = \varphi \circ \pi$$

and hence

$$(\pi \circ \tilde{\varphi})_{*|u}(B(\xi)|_u) = (\varphi \circ \pi)_{*|u}(B(\xi)|_u)$$

$$= \varphi_{*|p}u(\xi) = \tilde{\varphi}(u)(\xi).$$

The uniqueness of $B(\xi)$ leads to

$$\tilde{\varphi}_{*|u}(B(\xi)|_u) = B(\xi)|_{\tilde{\varphi}(u)}. \tag{1.23}$$

The required result follows then from (1.22), (1.23) and the definition of g_∇.

Finally we need a well-known result of the theory of Lie groups. This result follows from [39, Theorem VIII, chapter IV, p. 105]. Here we give a modified proof.

PROPOSITION 1.9. *Let* M *be a connected and simply connected manifold of dimension* n. *Further let* $X_1,\ldots,X_n$ *be* n *vector fields such that*

i) $X_1,\ldots,X_n$ *are complete;*

ii) $X_1,\ldots,X_n$ *are linearly independent at each point of* M (they determine an absolute parallelism);

iii) $[X_i,X_j] = \sum_k c_{ij}^k X_k$, *where the* c_{ij}^k *are constant.*

Then, for a fixed point $p \in M$, *the manifold* M *has a unique Lie group structure such that* p *is the identity and such that the vector fields* X_i *are all left invariant.* (They constitute a basis for the Lie algebra of the Lie group.)

Proof. Let $\tilde{\nabla}$ denote the linear connection on M defined by

$$\tilde{\nabla}_{X_i} X_j = 0 , \qquad i,j = 1,\ldots,n. \tag{1.24}$$

The curvature of $\tilde{\nabla}$ vanishes and the torsion of $\tilde{\nabla}$ is parallel because