Introduction

1. Fluid dynamics

There are two main reasons for studying fluid dynamics. Firstly, understanding (though in some places still only partial) can be gained of a great range of phenomena, many of which are of considerable complexity. And secondly, predictions can be made in many areas of practical importance which involve fluids.

As we shall see later, a 'fluid' is a way of looking at a large collection of particles, so as to avoid dealing with each particle separately. One of the largest examples of such a collection of 'particles' is a galaxy, composed of a vast number of individual stars. A more obvious fluid composes the sun: the particles here are largely electrons and nucleons, and the fluid dynamics here is complicated by electromagnetic forces, nuclear reactions and radiation effects. Astrophysics provides another example of fluid motion in the solar wind, the outflow of (isolated) particles from the sun: this is a fluid in which the particles are very thinly spread, but it is a fluid which interacts importantly with the Earth's magnetic field and the upper layers of the atmosphere. Both atmosphere and magnetic field are much studied examples of fluid dynamics; climate predictions, weather forecasts and studies of local climate are of obvious interest, while the origin of the Earth's magnetic field in the inner motions of the Earth's material is rather less obvious, but no less interesting. You may list for yourself some of the physical phenomena associated with the existence of oceans, rivers, lakes and underground water.

At this sort of scale one can start to see practical considerations coming to the fore. The engineer who designs a hydroelectric system must have a good knowledge of how water will behave and what forces it will exert. As a further example, an aeroplane (or a ship) is built by an interaction of past experience, mathematical calculation and testing of models in wind tunnels (or wave tanks). High speed trains, and suspension bridges, are in constant interaction with the wind. Even a low speed bicycle rides on a few thin films of lubricating oil.

Man himself can be regarded as a collection of fluid dynamic systems (a rather restricted view), and physiological fluid dynamics has recently emerged as an important area of research.

The above examples of areas involving the motion of fluids could easily be added to almost indefinitely: you should, for example, list all the fluid dynamic aspects of a petrol engined car. But these examples will serve to show the range and applicability of fluid dynamics. Moreover, the range and the applications increase, because fluid dynamics is an active field of research, not only in universities but also in industrial research associations and in national research centres.

Fluid dynamics is a branch of applied mathematics; the subject cannot be studied to any depth without a considerable skill in mathematics. This is one of its fascinations for any mathematically inclined person, to see how much of the apparatus of mathematics is needed to describe such a 'simple' problem as the flow of a fluid past an obstacle. In fact, problems in fluid dynamics have caused developments in mathematical techniques; the idea of a boundary layer (Chapter IX, §3) has stimulated the growth of the mathematics of 'singular perturbation theory', to give one example described in this text.

The application of mathematics to problems in the real world of physics and engineering is a skill that is hard to learn. Any real problem has too many aspects for us to hope to describe them all mathematically at once – if a man drops a book in a room, the air flow that is generated must depend on the shape of the room and the position and shape of all the objects in it: how can we solve a problem as complicated as that? The art is to describe the non-essential parts of the problem one by one until a mathematical problem can be formulated that is easy enough to solve, but still contains the essence of the original situation. The fluid dynamics in this text provides many examples of this reduction of reality to a simple model which can be treated mathematically and which shows the nature of the phenomenon under discussion. We may hope that after an intelligent study of these examples you will be ready to try your hand at 'mathematical modelling' of this kind.

2. Structure of the text

This text cannot start at the beginning, or it would become far too long. We must assume that you already know something about mathematics, about physics, and about the world around you and how to describe it mathematically. The first three chapters (all pleasantly short) are to remind you about some important things and, if necessary, to introduce you to others. All this material is, of course, important; but perhaps the mathematics for vector fields contained in Chapter I is the most important: you certainly cannot get by without it in later work.

The next two chapters (IV and V) are about the description of the velocities of a fluid. This is 'just' kinematics, there is as yet no discussion of the forces acting or of an equation of motion to predict what will result from a given initial situation. Some of the basic models and concepts are brought in at this stage, and the mathematics of

$$\nabla\cdot \text{ and } \nabla\times$$

or

$$\text{div and curl}$$

is needed.

The next three chapters (VI – VIII) discuss the forces acting in a fluid, and lead up to a full equation of motion for a fluid. All three chapters are concerned with the idea of pressure in a fluid. Chapter VI introduces pressure and deals with the easiest case, that of the pressure in a fluid at rest, and the forces it causes. Chapter VII discusses the possible relations between pressure and density, and outlines some necessary thermodynamics. Chapter VIII derives the relation between pressure and acceleration in a fluid, and discusses how it might be simplified in commonly occurring situations. Some of the derivations of equations in this part of the text are quite complicated, and need not be mastered at a first reading; but it would be a pity not to look through them, at least.

By the end of Chapter VIII there is nothing left to do except solve the equation of motion! This 'nothing' is a very large amount of work, because the equations of fluid dynamics are impossibly hard in their full generality. So we set off in Chapter IX with some moderately easy flows and using almost the full equations; these are flows in which the viscosity of the fluid is important, and they give valuable ideas on when viscosity must be included, and what some of its effects will be. Chapter X continues this line of thought by discussing flows for which viscosity can mainly be neglected, and Chapter XI is about a smaller class of rather simple flows which have a very easy (relatively!) equation to describe them and which are sometimes realistic. This group of three chapters concludes the most basic course on fluid dynamics: you can pick and choose from other chapters, but you must cover (even if not in all the details) Chapters I–XI.

The two chapters on sound and water waves are about small disturbances to a state of rest, with simple boundaries. They provide interesting preliminary descriptions of a number of obvious phenomena, without being too hard. The mathematics is mainly linear, and so comparatively easy.

The two chapters after that provide an introduction to what happens when disturbances are not small. Chapter XIV is about some of the effects of compressibility, and shows that discontinuities ('shock waves') can appear in the solutions of the apparently appropriate equations. Chapter XV uses some rather similar mathematics to deal with water waves of larger size on shallow water. Because much of the mathematics is similar, these two chapters go together, much as the previous two do.

The last two chapters use some advanced mathematics to discuss an approximate version of the fluid dynamics of aeroplane wings. You should not be unduly put off by the need for complex function theory here – not too much is needed, and you may find it easier than you expect. These two chapters are independent of the previous four.

Generally speaking, the mathematical techniques you need for this text increase as you go through it, and extra methods are not brought in until they are absolutely necessary. This is often done by postponing particular examples until the required methods have been explained, at the point when they are absolutely essential. Brief versions of the new mathematics are given in this text; if you need more, you must seek help in books on mathematical techniques or advanced calculus.

3. Method of working

If you try to study fluid dynamics purely as a branch of mathematics, then you are liable to get answers which do not agree with experiment or observation. This is because there always has to be a careful choice of the mathematical model that is to be used to describe a particular phenomenon: injudicious modelling will retain the wrong terms in the equations, and so not give a description of what you want. The test of the mathematical model must always be against reality.

It is only rarely that a student has the time or the opportunity to carry out serious experiments in fluid dynamics for himself. So a text has to provide some sort of substitute. What is recommended here is the study of films of carefully devised experiments: these give a better idea of the motions involved than any (reasonable) number of still photographs, even though these can be quite revealing. There is a good set of film loops, each lasting only a few minutes and each devoted to a single idea; and there are some longer films available which cover whole areas. You should spend some time with these films or loops as each new area of theory comes along.

Moreover, you should spend time looking around you. Examples of fluid dynamics are everywhere, and you should try to relate what you see to the topics you are studying. Sometimes you *can* do simple experiments, or use the weather maps in newspapers as a source of observational data. But do not expect too much; this is only a first text on fluid dynamics, and so the theories will be rather too simple to give more than a roughly correct answer.

The models we shall use in this text will be as simple as possible, and even so the mathematics may be felt to be rather hard. You must study the models carefully to see not only how the mathematics is operated, but also how the physical reality has been modelled. There are many examples in the text to help you along.

Next comes the real test. Can you do the problems at the end of each chapter? If not, ask yourself what section of the text it is about, and return there for guidance or inspiration. If that fails, try the hints at the back of the text, and start again.

No text book is ever perfect for everyone; you may need other books to give a different explanation of some points, to give some other examples, or to give some photographs for study. Or you may want to find a book to cover some application for which there is no room here. At the end of each chapter you will find some references to help with any of these quests, and also suggestions for books at a higher level for when you feel you are ready. But do not feel guilty if you never look at other texts; you will have quite enough to do working through this one.

Reference

The film loops and films referred to above are fully described in *Illustrated Experiments in Fluid Dynamics*, National Committee for Fluid Mechanics Films, 1972.

They are distributed in Great Britain by Fergus Davidson Associates Ltd, 376 London Road, West Croydon, Surrey CRO 2SU on behalf of Encyclopaedia Brittanica Educational Corp, 425 N. Michigan Ave, Chicago, Ill 60611.

If you have no opportunity to see films or film loops, then you should spend some time looking at photographs in other texts.

1

Mathematical preliminaries

1. Background knowledge

Quite a lot of mathematics is needed for this text, and most of it must be assumed to have been met elsewhere. However, it would be unfair to assume that you have already learned every mathematical technique, so some new methods will be introduced in sufficient detail as they become necessary. If you need to revise any of the topics mentioned below, do it now, before your lack of knowledge interferes with the fluid dynamics that is being expounded.

(a) *Vectors*

Fluid dynamics is about the motion of fluids, so velocities must come in; that is, the whole subject will be full of work with vectors. You must be confident in your use of the two products

$$\mathbf{A}\cdot\mathbf{B} \text{ and } \mathbf{A}\times\mathbf{B},$$

and in the use of components and unit vectors. These components may be those appropriate to cartesian axes or to polar directions; the important material about polar coordinates and directions is summarised below.

(b) *Functions of several variables*

The velocities in fluid dynamics will in general depend on position and time, e.g.

$$\mathbf{v}(x, y, z, t).$$

For an example of this you can think of weather maps, which usually show wind velocities at the surface. These velocities are different for different parts of the country and change from day to day. So we shall certainly need to consider functions of several variables; whenever possible we shall reduce from the generality of four variables down to special cases involving only two, but we cannot often use only one variable.

(c) Vector calculus

Since the velocities change in space and time, we shall need vector calculus. For example, the divergence of $\mathbf{v}$

$$\nabla \cdot \mathbf{v} \text{ or } \operatorname{div} \mathbf{v}$$

and the curl of $\mathbf{v}$

$$\nabla \times \mathbf{v} \text{ or } \operatorname{curl} \mathbf{v}$$

are most important. This whole area is vital, and so it is summarised below with special results that are needed in fluid dynamics. If you are not happy with the operations of vector calculus, then some of the work in later chapters will be almost impossible: you should learn now what is needed later, either from the material below or from some other text, such as the ones quoted in the list of reference books.

(d) Tensor notation

Some of the manipulations of vector calculus are most easily carried out by using the suffix notation of cartesian tensors. For example, the vector $\mathbf{v}$ is represented by its ith component v_i in many calculations. There are two second order tensors which have to come into later chapters, and so it is necessary to have some background in this area. The material is summarized below, and references are given at the end of this chapter if you need to start from scratch.

(e) Differential equations

Because fluid dynamics is a branch of dynamics, we shall have differential equations expressing the relation

$$\text{force} = \text{mass} \times \text{acceleration}.$$

Basic methods for ordinary differential equations should be known before you start. It would be a help if you had met some methods for simple first and second order partial differential equations; the necessary material will be presented in the text, but it is easier to get a working knowledge of the techniques if you have met them some time before you find that you have to apply them.

In particular, the method of characteristics will be important in Chapters XIV and XV; and the method of separation of variables is needed from Chapter XI on. The wave equation

$$c^2\,\partial^2 u/\partial x^2 = \partial^2 u/\partial t^2$$

forms the basis for much of the work described in Chapters XII and XIII; Laplace's equation

$$\nabla^2\phi = 0$$

is needed in Chapter XI; and the diffusion equation

$$k\,\partial^2 u/\partial x^2 = \partial u/\partial t$$

occurs in Chapter IX. The more you know about these three equations, the better. For all these methods and equations the necessary material appears in the text, but it would be a considerable advantage if that was not your first meeting with the ideas and methods.

(*f*) *Fourier series*

It is often found to be convenient to represent the solutions of partial differential equations in terms of series based on eigenfunctions of some ordinary differential equation. If this is not to seem a totally strange method to you, then an acquaintance with Fourier series will be needed. All that is needed is the techniques, we shall not need theorems. Other examples of eigenfunction expansions will come in, from Chapter XI on. If you already know about Legendre polynomials and Bessel functions, so much the better. But you should be able to manage by using the explanations given when these functions arise naturally in solving particular problems in fluid dynamics.

(*g*) *Complex numbers and functions*

Some complex function theory is needed in the last two chapters, and this is not described in this text – if you intend to work on these chapters, you must find a text on functions of a complex variable to help you. But for the two chapters (XII and XIII) on small amplitude waves you will need to be familiar with complex numbers and the formula

$$\cos kx + i \sin kx = e^{ikx}.$$

This will be used to replace the rather clumsy cosine and sine functions with the easier exponential function. We assume that you are already familiar with the algebra of complex numbers, including

$$z = |z|e^{i\theta}$$

and

$$z = \mathscr{R}e\, z + i\mathscr{I}m\, z.$$

2. Polar coordinate systems

We start with a careful revision of some results for plane polar coordinates, and then go on to use the same methods for the three-dimensional cylindrical polar and spherical polar systems.

(a) *Plane polar coordinates*

Plane polar coordinates (r, θ) are related to cartesian coordinates (x, y) by (see fig. I.1)

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \end{cases}$$

and the unit vectors $\mathbf{i}, \mathbf{j}$ of cartesians are related to those for polars, $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$, by

$$\begin{cases} \hat{\mathbf{r}} = \mathbf{i}\cos\theta + \mathbf{j}\sin\theta, \\ \hat{\boldsymbol{\theta}} = -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta. \end{cases}$$

If you calculate $\partial\hat{\mathbf{r}}/\partial\theta$ and $\partial\hat{\boldsymbol{\theta}}/\partial\theta$ from these formulae, you get

$$\begin{cases} \partial\hat{\mathbf{r}}/\partial\theta = \hat{\boldsymbol{\theta}}, \\ \partial\hat{\boldsymbol{\theta}}/\partial\theta = -\hat{\mathbf{r}}. \end{cases}$$

Clearly also

$$\partial\hat{\mathbf{r}}/\partial r = \partial\hat{\boldsymbol{\theta}}/\partial r = 0.$$

Now suppose that you allow small changes dr and $d\theta$ in the formulae for x and y. The corresponding small changes dx and dy are given by

$$\begin{cases} dx = dr\cos\theta - r\sin\theta\, d\theta, \\ dy = dr\sin\theta + r\cos\theta\, d\theta. \end{cases}$$

Fig. I.1. Plane polar coordinates.