CHAPTER 1 THE KORTEWEG–DE VRIES EQUATION

1 *The discovery of solitary waves*

This book is an introduction to the theory of solitons and to the applications of the theory. Solitons are a special kind of localized wave, an essentially nonlinear kind. We shall define them at the end of this chapter, describing their discovery by Zabusky & Kruskal (1965). A *solitary wave* is the first and most celebrated example of a soliton to have been discovered, although more than 150 years elapsed after the discovery before a solitary wave was recognized as an example of a soliton. To lead to the definition of a soliton, it is helpful to study solitary waves on shallow water. We shall describe briefly in this section the properties of these waves, and then revise the elements of the theory of linear and nonlinear waves in order to build a foundation of the theory of solitons. Let us begin at the beginning, and relate a little history.

The solitary wave, or great wave of translation, was first observed on the Edinburgh to Glasgow canal in 1834 by J. Scott Russell. Russell reported his discovery to the British Association in 1844 as follows:

> I believe I shall best introduce this phænomenon by describing the circumstances of my own first acquaintance with it. I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped — not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height

> gradually diminished, and after a chase of one or two miles
> I lost it in the windings of the channel.

Russell also did some laboratory experiments, generating solitary waves by dropping a weight at one end of a water channel (see Fig. 1). He deduced empirically that the volume of water in the wave is equal to the volume displaced by the weight and that the steady velocity c of the wave is given by

$$c^2 = g(h + a), \tag{1}$$

where the amplitude a of the wave and the height h of the undisturbed water are as defined in Fig. 2. Note that a taller solitary wave travels faster than a smaller one. Russell also made many other observations and experiments on solitary waves. In particular, he tried to generate waves of depression by raising the weight from the bottom of the channel initially. He found, however, that an initial depression becomes a train of oscillatory waves whose lengths increase and amplitudes decrease with time (see Fig. 3).

Boussinesq (1871) and Rayleigh (1876) independently assumed that a solitary wave has a length much greater than the depth of the water

Fig. 1. Russell's solitary wave: a diagram of its development. (a) The start. (b) Later. (After Russell 1844.)

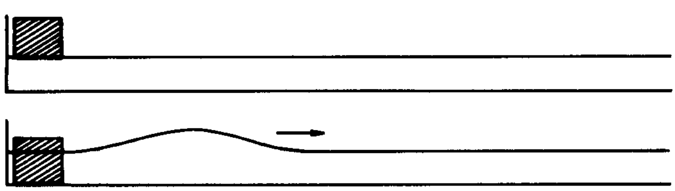
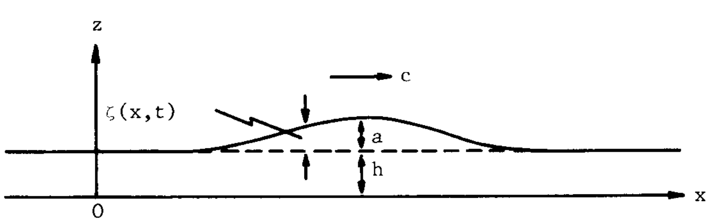


Fig. 2. The configuration and parameters for description of a solitary wave.



and thereby deduced Russell's empirical formula for c from the equations of motion of an inviscid incompressible fluid. They further showed essentially that the wave height above the mean level h is given by

$$\zeta(x,t) = a\,\mathrm{sech}^2\{(x - ct)/b\}, \qquad (2)$$

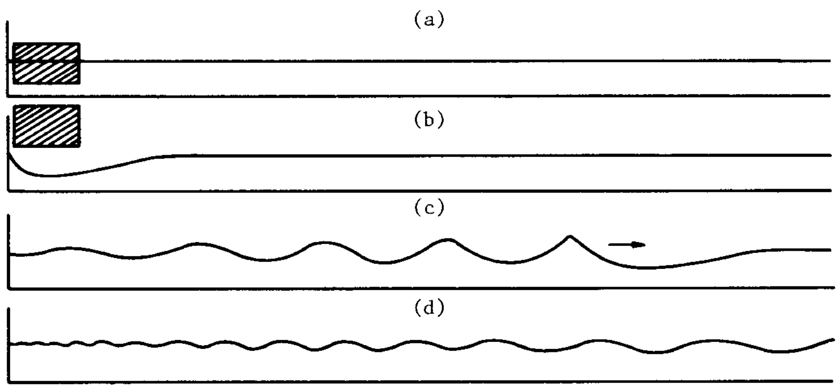
where $b^2 = 4h^2(h + a)/3a$ for any positive amplitude a.

In 1895 Korteweg and de Vries developed this theory, and found an equation governing the two-dimensional motion of weakly nonlinear long waves:

$$\frac{\partial\zeta}{\partial t} = \frac{3}{2}\sqrt{\frac{g}{h}}\left(\zeta\frac{\partial\zeta}{\partial x} + \frac{2}{3}\alpha\frac{\partial\zeta}{\partial x} + \frac{1}{3}\sigma\frac{\partial^3\zeta}{\partial x^3}\right), \qquad (3)$$

where α is a small but otherwise arbitrary constant, $\sigma = \frac{1}{3}h^3 - Th/g\rho$, and T is the surface tension of the liquid of density ρ. This is essentially the original form of the *Korteweg-de Vries equation*; we shall call it the *KdV equation*. Note that in the approximations used to derive this equation one considers long waves propagating in the direction of increasing x. A similar equation, with $-\partial\zeta/\partial t$ instead of $\partial\zeta/\partial t$, may be applied to waves propagating in the opposite direction.

Fig. 3. Russell's observations of oscillating waves: successive stages of their development. (After Russell 1844.)



2 *Fundamental ideas*

We shall dwell on the mathematical ideas, rather than the applications to water waves, in this book. First note that by translations and magnifications of the dependent and independent variables,

$$u = k_1\zeta + k_o, \quad X = k_3x + k_2, \quad T = k_4t + k_5, \tag{1}$$

we can write the KdV equation in many equivalent forms by choice of the constants k_o to k_5:

$$\frac{\partial u}{\partial T} = (1 + u)\frac{\partial u}{\partial X} + \frac{\partial^3 u}{\partial X^3}, \tag{2}$$

$$\frac{\partial u}{\partial T} + (1 + u)\frac{\partial u}{\partial X} + \frac{\partial^3 u}{\partial X^3} = 0, \tag{3}$$

$$\frac{\partial u}{\partial T} - 6u\frac{\partial u}{\partial X} + \frac{\partial^3 u}{\partial X^3} = 0, \tag{4}$$

$$\frac{\partial u}{\partial T} + 6u\frac{\partial u}{\partial X} + \frac{\partial^3 u}{\partial X^3} = 0, \text{ etc.} \tag{5}$$

(These transformations are examples of *Lie groups* or *continuous groups*, which are the subject of an extensive theory and which have many applications, notably in physics. For further reading, the book by Bluman & Cole (1974) is recommended.) We shall usually use equation (4) as the standard form.

To understand how solitons may persist, take the KdV equation in the form

$$\frac{\partial u}{\partial t} + (1 + u)\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \tag{6}$$

and seek the properties of small-amplitude waves. Accordingly, linearize the equation to get

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \tag{7}$$

approximately. For this linearized equation any solution can be represented as a superposition of Fourier components. So we use the method of normal modes, with independent components $u \propto e^{i(kx-\omega t)}$. It follows

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that

$$\omega = k - k^3. \tag{8}$$

This is the *dispersion relation* which gives the frequency ω as a function of the wavenumber k. From it we deduce the *phase velocity*,

$$c = \frac{\omega}{k} = 1 - k^2, \tag{9}$$

which gives the velocity of the wave fronts of the sinusoidal mode. We also deduce the *group velocity*,

$$c_g = \frac{d\omega}{dk} = 1 - 3k^2, \tag{10}$$

which gives the velocity of a wave packet, i.e. a group of waves with nearly the same length $2\pi/k$. Note that $c_g \leqslant c \leqslant 1$, and $c = c_g = 1$ for long waves (i.e. for $k = 0$). Also a short wave has a negative phase velocity c.

Packets of waves of nearly the same length propagate with the group velocity, individual components moving through the packet with their phase velocity. It can in general be shown that the energy of a wave disturbance is propagated at the group velocity, not the phase velocity. Long-wave components of a general solution travel faster than the short-wave components, and thereby the components disperse. Thus the linear theory predicts the dispersal of any disturbance other than a purely sinusoidal one. Looking back to the equation, you can see that the dispersion comes from the term in k^3 in the expression for ω and thence from the term $\partial^3 u/\partial x^3$ in the KdV equation.

For further reading on group velocity, Lighthill's (1978, §3.6) book is recommended.

In contrast to dispersion, nonlinearity leads to the concentration of a disturbance. To see this, neglect the term $\partial^3 u/\partial x^3$ in the KdV equation above and retain the nonlinear term. Then we have

$$\frac{\partial u}{\partial t} + (1 + u)\frac{\partial u}{\partial x} = 0. \tag{11}$$

The method of characteristics may be used to show that this equation has the elementary solution

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$$u = f\{x - (1 + u)t\} \tag{12}$$

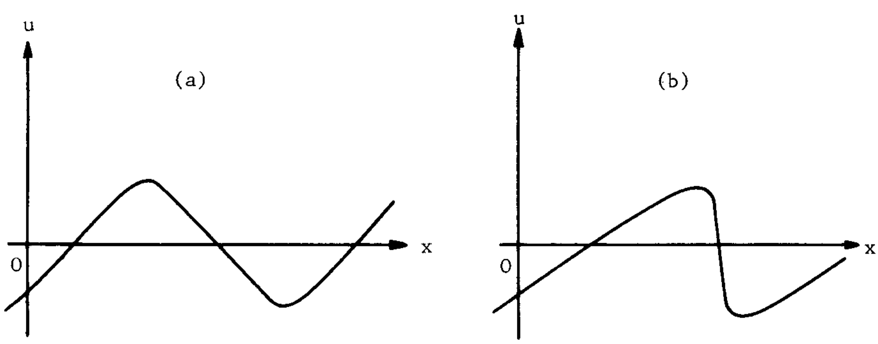
for any differentiable function f. (You may also verify that this satisfies equation (11).) This shows that disturbances travel at the characteristic velocity $1 + u$. Thus the 'higher' parts of the solution travel faster than the 'lower'. This 'catching up' tends to steepen a disturbance until it 'breaks' and a discontinuity or *shock wave* forms (see Fig. 4).

For further reading on wave breaking, the books by Landau & Lifshitz (1959, §94) and Whitham (1974, §2.1) are recommended.

We anticipate that for a solitary wave the dispersive effects of the term $\partial^3u/\partial x^3$ and the concentrating effects of the term $u\partial u/\partial x$ are just in balance. We shall examine the details of this balance in the next few chapters. A similar balance occurs for a large number of solutions of nonlinear equations, a few of which will be demonstrated in the text and problems of this book.

In future, we shall usually denote partial differentiation by a subscript, so that $u_t = \partial u/\partial t$, $u_x = \partial u/\partial x$, $u_{xxx} = \partial^3u/\partial x^3$ etc.

Fig. 4. Sketch of the nonlinear steepening of a wave as it develops. (a) t = 0. (b) Later.



3 *The discovery of soliton interactions*

Examining the Fermi-Pasta-Ulam model of phonons in an anharmonic lattice, Zabusky & Kruskal (1965) were led to work on the KdV equation. They considered the following initial-value problem in a periodic domain:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \delta\frac{\partial^3 u}{\partial x^3} = 0 \tag{1}$$

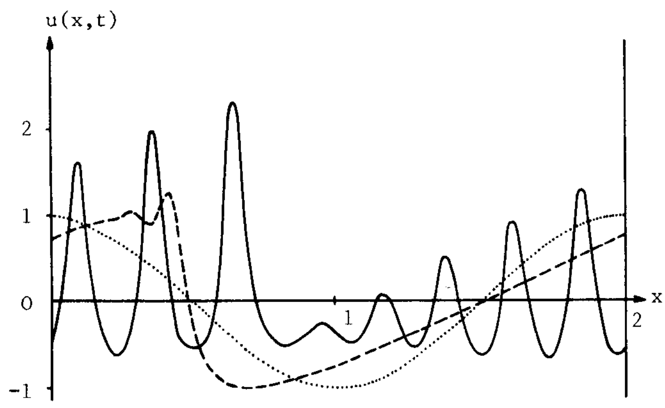
i.e. $u_t + uu_x + \delta u_{xxx} = 0,$

where

$$u(2,t) = u(0,t),\quad u_x(2,t) = u_x(0,t),\quad u_{xx}(2,t) = u_{xx}(0,t) \quad \text{for all } t, \tag{2}$$

and $u(x,0) = \cos\pi x \quad \text{for} \quad 0 \leqslant x \leqslant 2.$ (3)

Fig. 5. Solutions of the KdV equation $u_t + uu_x + \delta u_{xxx} = 0$ with $\delta = 0.022$ and $u(x,0) = \cos\pi x$ for $0 \leqslant x \leqslant 2$. (After Zabusky & Kruskal 1965.) (a) The dotted curve gives u at $t = 0$. (b) The broken curve gives u at $t = 1/\pi$. Note that a 'shock wave' has nearly formed at $x = 0.5$; this is because the initial solution is not steep enough for dispersion to be significant (i.e. because $|\delta u_{xxx}| << |uu_x|$ mostly) and therefore the terms $u_t + uu_x$ have been approximately zero up till this time. (c) The continuous curve gives u at $t = 3.6/\pi$. Note the formation of eight more-or-less distinct solitons, whose crests lie close to a straight line (extended with the period 2).



The periodic boundary conditions suit numerical integration of the system. Putting $\delta = 0.022$, Zabusky & Kruskal computed u for $t > 0$. They found that the solution breaks up into a train of eight solitary waves (see Fig. 5), each like a sech-squared solution, that these waves move through one another as the faster ones catch up the slower ones, and that finally the initial state (or something very close to it) recurs. This remarkable numerical discovery, that strongly nonlinear waves can interact and carry on thereafter almost as if they had never interacted, led to an intense study of the analytic and numerical properties of many kinds of *solitons*. This intense study continues still.

A 'soliton' is not precisely defined, but is used to describe any solution of a nonlinear equation or system which (i) represents a wave of permanent form; (ii) is localized, decaying or becoming constant at infinity; and (iii) may interact strongly with other solitons so that after the interaction it retains its form, almost as if the principle of superposition were valid. The word 'soliton' was coined by Zabusky & Kruskal (1965) after 'photon', 'proton', etc. to emphasize that a soliton is a localized entity which may keep its identity after an interaction. (The Greek word 'on' means 'solitary'.) The word may also symbolize the hope that the properties of elementary particles will be deduced by calculation of soliton solutions of some nonlinear field theory.

A solitary wave may be defined more generally than as a sech-squared solution of the KdV equation. We take it to be any solution of a nonlinear system which represents a hump-shaped wave of permanent form, whether it is a soliton or not.

4 *Applications of the KdV equation*

We have related how the KdV equation was discovered in 1895 to model the behaviour of weakly nonlinear long water waves. Benney (1966) recognized that this approximation, whereby a small quadratic term representing convection in a moving medium balances a linear term representing dispersion of long waves, is widely applicable (see also Problem 1.8). He applied it to inertial waves in a rotating fluid and to internal gravity waves in a stratified fluid. Two of the many other applications of the KdV equation are to ion-acoustic waves in a plasma (Washimi & Taniuti 1966) and to pressure waves in a misture of gas bubbles and liquid (Wijngaarden 1968).

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Problems

1.1 *Motion pictures of soliton interactions.* See the animated computer films of solitons by Zabusky, Kruskal & Deem (F1965) and Eilbeck (F1981). These films are listed in the motion picture index.

1.2 *The 'tail' of a solitary wave.* Verify that $u(x,t) = -2\kappa^2\text{sech}^2\{\kappa(x - 4\kappa^2 t)\}$ satisfies the KdV equation in the form

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0,$$

i.e. $$u_t - 6uu_x + u_{xxx} = 0.$$

Show that $u(x,t) = e^{2\kappa(x-ct)}$ is a solution of the linearized KdV equation,

$$u_t + u_{xxx} = 0,$$

if $c = 4\kappa^2$. How is the latter solution related to the former as $x \to -\infty$?

1.3 *The formation of a 'shock wave'.* Verify that if $u = \cos\{\pi(x - ut)\}$ then

$$u_t + uu_x = 0 \quad \text{for} \quad 0 \leqslant x < 2,$$

$u(x + 2,t) = u(x,t)$ for $t > 0$, and $u(x,0) = \cos\pi x$. Where (i.e. at what stations x) might u_x approach infinity? Show that u first ceases to be single valued when $t = 1/\pi$. How might one seek to continue the solution for $t > 1/\pi$?

1.4 *The initial-value problem for the linearized KdV equation.* If

$$u_t + u_{xxx} = 0,$$

$u(x,0) = g(x)$, and $u, u_x, u_{xx} \to 0$ as $x \to \pm\infty$, show that

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\{ik(x + k^2 t)\}\int_{-\infty}^{\infty} g(y)e^{-iky}dy\, dk$$

$$= (3t)^{-1/3}\int_{-\infty}^{\infty} g(y)\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left[i\left\{\frac{\alpha(x-y)}{(3t)^{1/3}} + \frac{1}{3}\alpha^3\right\}\right]d\alpha dy.$$

Deduce that

$$u(x,t) \sim \int_{-\infty}^{\infty} g(y)dy(3t)^{-1/3}Ai(X) - \int_{-\infty}^{\infty} yg(y)dy(3t)^{-2/3}Ai'(X) + \ldots$$

as $t \to \infty$ for fixed $X = x/(3t)^{1/3}$, where Ai is the Airy function. Show that the first term of the above asymptotic expansion represents a steeply rising wave front where $x \approx t^{1/3}$ and a slowly decaying wave train where $x \lesssim - t^{1/3}$.
[You are given that $Ai(X) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left\{i(\alpha X + \frac{1}{3}\alpha^3)\right\}d\alpha$.]

1.5 *The essential nonlinearity of a solitary wave.* Why is it that a solitary wave of infinitesimal amplitude persists whereas all localized solutions of the linearized KdV equation disperse and decay in time? Discuss.

1.6 *Similarity solutions of the KdV equation.* Show that the KdV equation,

$$u_t - 6uu_x + u_{xxx} = 0,$$

is invariant under the one-parameter continuous group of transformations $x \to kx$, $t \to k^3t$ and $u \to k^{-2}u$. Deduce that $t^{2/3}u$ and $x/t^{1/3}$ are also invariant under this group.

Assuming that there exists a solution of the form

$$u(x,t) = -(3t)^{-2/3}U(X), \text{ where } X = x/(3t)^{1/3},$$

show that

$$\frac{d^3U}{dX^3} + (U - X)\frac{dU}{dX} - 2U = 0.$$

[Miura (1976, p.437).]