1: Direct sums and Jordan forms

In this chapter we take as a central theme the notion of the *direct sum* $A \oplus B$ of subspaces A, B of a vector space V. Recall that $V = A \oplus B$ if and only if every $x \in V$ can be expressed uniquely in the form $a + b$ where $a \in A$ and $b \in B$; equivalently, if $V = A + B$ and $A \cap B = \{0\}$. For every subspace A of V there is a subspace B of V such that $V = A \oplus B$. In the case where V is of finite dimension, this is easily seen; take a basis $\{v_1, \ldots, v_k\}$ of A, extend it to a basis $\{v_1, \ldots, v_n\}$ of V, then note that $\{v_{k+1}, \ldots, v_n\}$ spans a subspace B such that $V = A \oplus B$.

If $f : V \to V$ is a linear transformation then a subspace W of V is said to be *f–invariant* if f maps W into itself. If W is f–invariant then there is an ordered basis of V with respect to which the matrix of V is of the form

$$\begin{bmatrix} M & N \\ 0 & X \end{bmatrix}$$

where M is of size $\dim W \times \dim W$.

If $f : V \to V$ is such that $f \circ f = f$ then f is called a *projection*. For such a linear transformation we have $V = \operatorname{Im} f \oplus \operatorname{Ker} f$ where the subspace $\operatorname{Im} f$ is f–invariant (and the subspace $\operatorname{Ker} f$ is trivially so). A vector space V is the direct sum of subspaces $W_1, \ldots, W_k$ if and only if there are non-zero projections $p_1, \ldots, p_k : V \to V$ such that

$$\sum_{i=1}^{k} p_i = \mathrm{id}_V \qquad \text{and} \qquad p_i \circ p_j = 0 \quad \text{for} \quad i \neq j.$$

In this case $W_i = \operatorname{Im} p_i$ for each i, and relative to given ordered bases of

$W_1, \ldots, W_k$ the matrix of f is of the diagonal block form

$$\begin{bmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_k \end{bmatrix}.$$

Of particular importance is the situation where each M_i is of the form

$$M_i = \begin{bmatrix} \lambda & 1 & 0 & \ldots & 0 & 0 \\ 0 & \lambda & 1 & \ldots & 0 & 0 \\ 0 & 0 & \lambda & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \lambda & 1 \\ 0 & 0 & 0 & \ldots & 0 & \lambda \end{bmatrix}$$

in which case the diagonal block matrix is called a *Jordan matrix.*

The Cayley–Hamilton theorem says that a linear transformation f is a zero of its characteristic polynomial. The minimum polynomial of f is the monic polynomial of least degree of which f is a zero. When the minimum polynomial of f factorises into a product of linear polynomials then there is a basis of V with respect to which the matrix of f is a Jordan matrix. This matrix is unique (up to the sequence of the diagonal blocks), the diagonal entries λ above are the eigenvalues of f, and the number of M_i associated with a given λ is the geometric multiplicity of λ. The corresponding basis is called a *Jordan basis.*

We mention here that, for space considerations in the solutions, we shall often write an eigenvector

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

as $[x_1, x_2, \ldots, x_n]$.

1.1 Which of the following statements are true? For those that are false, give a counter-example.

(i) If $\{a_1, a_2, a_3\}$ is a basis for $\mathbb{R}^3$ and b is a non-zero vector in $\mathbb{R}$ then $\{b + a_1, a_2, a_3\}$ is also a basis for $\mathbb{R}^3$.

(ii) If A is a finite set of linearly independent vectors then the dimension of the subspace spanned by A is equal to the number of vectors in A.

(iii) The subspace $\{(x,x,x) \mid x \in \mathbb{R}\}$ of $\mathbb{R}^3$ has dimension 3.

(iv) If A is a linearly dependent set of vectors in $\mathbb{R}^n$ then there are more than n vectors in A.

(v) If A is a linearly dependent subset of $\mathbb{R}^n$ then the dimension of the subspace spanned by A is strictly less than the number of vectors in A.

(vi) If A is a subset of $\mathbb{R}^n$ and the subspace spanned by A is $\mathbb{R}^n$ itself then A contains exactly n vectors.

(vii) If A and B are subspaces of $\mathbb{R}^n$ then we can find a basis of $\mathbb{R}^n$ which contains a basis of A and a basis of B.

(viii) An n–dimensional vector space contains only finitely many subspaces.

(ix) If A is an $n \times n$ matrix over $\mathbb{Q}$ with $A^3 = I$ then A is non-singular.

(x) If A is an $n \times n$ matrix over $\mathbb{C}$ with $A^3 = I$ then A is non-singular.

(xi) An isomorphism between two vector spaces can always be represented by a square singular matrix.

(xii) Any two n–dimensional vector spaces are isomorphic.

(xiii) If A is an $n \times n$ matrix such that $A^2 = I$ then $A = I$.

(xiv) If A, B and C are non-zero matrices such that $AC = BC$ then $A = B$.

(xv) The identity map on $\mathbb{R}^n$ is represented by the identity matrix with respect to any basis of $\mathbb{R}^n$.

(xvi) Given any two bases of $\mathbb{R}^n$ there is an isomorphism from $\mathbb{R}^n$ to itself that maps one basis onto the other.

(xvii) If A and B represent linear transformations $f, g : \mathbb{R}^n \to \mathbb{R}^n$ with respect to the same basis then there is a non-singular matrix P such that $P^{-1}AP = B$.

(xviii) There is a bijection between the set of linear transformations from $\mathbb{R}^n$ to itself and the set of $n \times n$ matrices over $\mathbb{R}$.

(xix) The map $t : \mathbb{R}^2 \to \mathbb{R}^2$ given by $t(x,y) = (y, x+y)$ can be represented by the matrix

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

with respect to some basis of $\mathbb{R}^2$.

(xx) There is a non-singular matrix P such that $P^{-1}AP$ is diagonal for any non-singular matrix A.

1.2 Let $t_1, t_2, t_3, t_4 \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$ be given by

$$\begin{aligned} t_1(a,b,c) &= (a+b, b+c, c+a); \\ t_2(a,b,c) &= (a-b, b-c, 0); \\ t_3(a,b,c) &= (-b, a, c); \\ t_4(a,b,c) &= (a, b, b). \end{aligned}$$

Find $\operatorname{Ker} t_i$ and $\operatorname{Im} t_i$ for $i = 1,2,3,4$. Is it true that $\mathbb{R}^3 = \operatorname{Ker} t_i \oplus \operatorname{Im} t_i$ for any of $i = 1,2,3,4$?

Is $\operatorname{Im} t_2$ t_3–invariant? Is $\operatorname{Ker} t_2$ t_3–invariant?

Find $t_3 \circ t_4$ and $t_4 \circ t_3$. Compute the images and kernels of these composites.

1.3 Let V be a vector space of dimension 3 over a field F and let $t \in \mathcal{L}(V,V)$ be represented by the matrix

$$\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

with respect to some basis of V. Find $\dim \operatorname{Ker} t$ and $\dim \operatorname{Im} t$ when

(i) $F = \mathbb{R}$;
(ii) $F = \mathbb{Z}_2$;
(iii) $F = \mathbb{Z}_3$.

Is $V = \operatorname{Ker} t \oplus \operatorname{Im} t$ in any of cases (i), (ii) or (iii)?

1.4 Let V be a finite-dimensional vector space and let $s, t \in \mathcal{L}(V,V)$ be such that $s \circ t = \mathrm{id}_V$. Prove that $t \circ s = \mathrm{id}_V$. Prove also that a subspace of V is t–invariant if and only if it is s–invariant. Are these results true when V is infinite-dimensional?

1.5 Let V_n be the vector space of polynomials of degree less than n over the field $\mathbb{R}$. If $D \in \mathcal{L}(V_n, V_n)$ is the differentiation map, find $\operatorname{Im} D$ and $\operatorname{Ker} D$. Prove that $\operatorname{Im} D \simeq V_{n-1}$ and that $\operatorname{Ker} D \simeq \mathbb{R}$. Is it true that

$$V_n = \operatorname{Im} D \oplus \operatorname{Ker} D\ ?$$

Do the same results hold if the ground field $\mathbb{R}$ is replaced by the field $\mathbb{Z}_2$?

1.6 Let V be a finite-dimensional vector space and let $t \in \mathcal{L}(V,V)$. Establish the chains

$$V \supseteq \operatorname{Im} t \supseteq \operatorname{Im} t^2 \supseteq \ldots \supseteq \operatorname{Im} t^n \supseteq \operatorname{Im} t^{n+1} \supseteq \ldots;$$
$$\{0\} \subseteq \operatorname{Ker} t \subseteq \operatorname{Ker} t^2 \subseteq \ldots \subseteq \operatorname{Ker} t^n \subseteq \operatorname{Ker} t^{n+1} \subseteq \ldots.$$

Show that there is a positive integer p such that $\operatorname{Im} t^p = \operatorname{Im} t^{p+1}$ and deduce that

$$(\forall k \geq 1) \qquad \operatorname{Im} t^p = \operatorname{Im} t^{p+k} \quad \text{and} \quad \operatorname{Ker} t^p = \operatorname{Ker} t^{p+k}.$$

Show also that

$$V = \operatorname{Im} t^p \oplus \operatorname{Ker} t^p$$

and that the subspaces $\operatorname{Im} t^p$ and $\operatorname{Ker} t^p$ are t–invariant.

1.7 Let V be a vector space of dimension n over a field F and let $f : V \to V$ be a non-zero linear transformation such that $f \circ f = 0$. Show that if $\operatorname{Im} f$ is of dimension r then $2r \leq n$. Suppose now that W is a subspace of V such that $V = \operatorname{Ker} f \oplus W$. Show that W is of dimension r and that if $\{w_1, \ldots, w_r\}$ is a basis of W then $\{f(w_1), \ldots, f(w_r)\}$ is a linearly independent subset of $\operatorname{Ker} f$. Deduce that $n-2r$ elements $x_1, \ldots, x_{n-2r}$ can be chosen in $\operatorname{Ker} f$ such that

$$\{w_1, \ldots, w_r, f(w_1), \ldots, f(w_r), x_1, \ldots, x_{n-2r}\}$$

is a basis of V.

Hence show that a non-zero $n \times n$ matrix A over F is such that $A^2 = 0$ if and only if A is similar to a matrix of the form

$$\begin{bmatrix} 0_r & 0 \\ I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

1.8 Let V be a vector space of dimension 4 over $\mathbb{R}$. Let a basis of V be $B = \{b_1, b_2, b_3, b_4\}$. Writing each $x \in V$ as $x = \sum_{i=1}^{4} x_i b_i$, let

$$V_1 = \{x \in V \mid x_3 = x_2 \text{ and } x_4 = x_1\},$$
$$V_2 = \{x \in V \mid x_3 = -x_2 \text{ and } x_4 = -x_1\}.$$

Show that

(1) V_1 and V_2 are subspaces of V;

(2) $\{b_1 + b_4, b_2 + b_3\}$ is a basis of V_1 and $\{b_1 - b_4, b_2 - b_3\}$ is a basis of V_2;

(3) $V = V_1 \oplus V_2$;

(4) with respect to the basis B and the basis

$$C = \{b_1 + b_4, b_2 + b_3, b_2 - b_3, b_1 - b_4\}$$

the matrix of id_V is

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & -\frac{1}{2} \end{bmatrix}.$$

A 4×4 matrix M over $\mathbb{R}$ is said to be *centro-symmetric* if

$$m_{ij} = m_{5-i,5-j}$$

for all i, j. If M is centro-symmetric, show that M is similar to a matrix of the form

$$\begin{bmatrix} \alpha & \beta & 0 & 0 \\ \gamma & \delta & 0 & 0 \\ 0 & 0 & \epsilon & \varsigma \\ 0 & 0 & \eta & \vartheta \end{bmatrix}.$$

1.9 Let V be a vector space of dimension n over a field F. Suppose first that F is not of characteristic 2 (i.e. that $1_F + 1_F \neq 0_F$). If $f : V \to V$ is a linear transformation such that $f \circ f = \mathrm{id}_V$ prove that

$$V = \mathrm{Im}(\mathrm{id}_V + f) \oplus \mathrm{Im}(\mathrm{id}_V - f).$$

Deduce that an $n \times n$ matrix A over F is such that $A^2 = I_n$ if and only if A is similar to a matrix of the form

$$\begin{bmatrix} I_p & 0 \\ 0 & -I_{n-p} \end{bmatrix}.$$

Suppose now that F is of characteristic 2 and that $f \circ f = \mathrm{id}_V$. If $g = \mathrm{id}_V + f$ show that

$$x \in \mathrm{Ker}\, g \iff x = f(x),$$

and that $g \circ g = 0$. Deduce that an $n \times n$ matrix A over F is such that $A^2 = I_n$ if and only if A is similar to a matrix of the form

$$\begin{bmatrix} I_{n-2p} & & & & \\ & \begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix} & & & \\ & & \begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix} & & \\ & & & \ddots & \\ & & & & \begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix} \end{bmatrix}.$$

[*Hint.* Observe that $\operatorname{Im} g \subseteq \operatorname{Ker} g$. Let $\{g(c_1), \dots, g(c_p)\}$ be a basis of $\operatorname{Im} g$ and extend this to a basis $\{b_1, \dots, b_{n-2p}, g(c_1), \dots, g(c_p)\}$ of $\operatorname{Ker} g$. Show that

$$\{b_1, \dots, b_{n-2p}, g(c_1), c_1, \dots, g(c_p), c_p\}$$

is a basis of V.]

1.10 Let V be a finite-dimensional vector space and let $t \in \mathcal{L}(V, V)$ be such that $t \neq \mathrm{id}_V$ and $t \neq 0$. Is it possible to have $\operatorname{Im} t \cap \operatorname{Ker} t \neq \{0\}$? Is it possible to have $\operatorname{Im} t = \operatorname{Ker} t$? Is it possible to have $\operatorname{Im} t \subset \operatorname{Ker} t$? Is it possible to have $\operatorname{Ker} t \subset \operatorname{Im} t$? Which of these are possible if t is a projection?

1.11 Is it possible to have projections $e, f \in \mathcal{L}(V, V)$ with $\operatorname{Ker} e = \operatorname{Ker} f$ and $\operatorname{Im} e \neq \operatorname{Im} f$? Is it possible to have $\operatorname{Im} e = \operatorname{Im} f$ and $\operatorname{Ker} e \neq \operatorname{Ker} f$? Is it possible to have projections e, f with $e \circ f = 0$ but $f \circ e \neq 0$?

1.12 Let V be a vector space over a field of characteristic not equal to 2. Let $e_1, e_2 \in \mathcal{L}(V, V)$ be projections. Prove that $e_1 + e_2$ is a projection if and only if $e_1 \circ e_2 = e_2 \circ e_1 = 0$.

If $e_1 + e_2$ is a projection, find $\operatorname{Im}(e_1 + e_2)$ and $\operatorname{Ker}(e_1 + e_2)$ in terms of the images and kernels of e_1, e_2.

1.13 Let V be the subspace of $\mathbb{R}^3$ given by

$$V = \{(a, a, 0) \mid a \in \mathbb{R}\}.$$

Find a subspace U of $\mathbb{R}^3$ such that $\mathbb{R}^3 = V \oplus U$. Is U unique? Find a projection $e \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$ such that $\operatorname{Im} e = V$ and $\operatorname{Ker} e = U$. Find also a projection $f \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$ such that $\operatorname{Im} f = U$ and $\operatorname{Ker} f = V$.

1.14 If V is a finite-dimensional vector space over a field F and $e, f \in \mathcal{L}(V,V)$ are projections prove that $\operatorname{Im} e = \operatorname{Im} f$ if and only if $e \circ f = f$ and $f \circ e = e$.

Suppose that $e_1, \dots, e_k \in \mathcal{L}(V,V)$ are projections with

$$\operatorname{Im} e_1 = \operatorname{Im} e_2 = \cdots = \operatorname{Im} e_k.$$

Let $\lambda_1, \lambda_2, \dots, \lambda_k \in F$ be such that $\sum_{i=1}^{k} \lambda_i = 1$. Prove that

$$e = \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_k e_k$$

is a projection with $\operatorname{Im} e = \operatorname{Im} e_i$.

Is it necessarily true that if $f_1, \dots, f_k \in \mathcal{L}(V,V)$ are projections and $\sum_{i=1}^{k} \lambda_i = 1$ then $\sum_{i=1}^{k} \lambda_i f_i$ is also a projection?

1.15 A *net* over the interval $[0,1]$ of $\mathbb{R}$ is a finite sequence $(a_i)_{0 \le i \le n+1}$ such that

$$0 = a_0 < a_1 < \cdots < a_n < a_{n+1} = 1.$$

A *step function* on $[0,1[$ is a mapping $f : [0,1[\to \mathbb{R}$ for which there exists a net $(a_i)_{0 \le i \le n+1}$ over $[0,1]$ and a finite sequence $(b_i)_{0 \le i \le n}$ of elements of $\mathbb{R}$ such that

$$(\forall x \in [a_i, a_{i+1}[) \qquad f(x) = b_i.$$

Show that the set E of step functions on $[0,1[$ is a vector space over $\mathbb{R}$ and that a basis of E is the set $\{e_k \mid k \in [0,1[\}$ of functions $e_k : [0,1[\to \mathbb{R}$ given by

$$e_k(x) = \begin{cases} 0 & \text{if } 0 \le x < k; \\ 1 & \text{if } k \le x < 1. \end{cases}$$

A *piecewise linear function* on $[0,1[$ is a mapping $f : [0,1[\to \mathbb{R}$ for which there exists a net $(a_i)_{0 \le i \le n+1}$ and sequences $(b_i)_{0 \le i \le n}, (c_i)_{0 \le i \le n}$ of elements of $\mathbb{R}$ such that

$$(\forall x \in [a_i, a_{i+1}[) \qquad f(x) = b_i x + c_i.$$

Let F be the set of piecewise linear functions on $[0,1[$ and let G be the subset of F consisting of the piecewise linear functions g that are continuous with $g(0) = 0$. Show that F, G are vector spaces over $\mathbb{R}$ and that $F = E \oplus G$.

Show that a basis of G is the set $\{g_k \mid k \in [0,1[\}$ of functions given by

$$g_k(x) = \begin{cases} 0 & \text{if } 0 \le x < k; \\ x - k & \text{if } k \le x < 1. \end{cases}$$

Finally, show that the assignment

$$f \longmapsto I(f) = \int_0^x f(t)\, dt$$

describes an isomorphism from E to G.

1.16 Let V be a vector space over a field F and let $t \in \mathcal{L}(V, V)$. Let λ_1 and λ_2 be distinct eigenvalues of t with associated eigenvectors v_1 and v_2. Is it possible for $\lambda_1 + \lambda_2$ to be an eigenvalue of t? What about $\lambda_1\lambda_2$?

1.17 Let $t \in \mathcal{L}(\mathbb{C}^2, \mathbb{C}^2)$ be given by

$$t(a, b) = (a + 2b, b - a).$$

Find the eigenvalues of t and show that there is a basis of $\mathbb{C}^2$ consisting of eigenvectors of t. Find such a basis, and the matrix of t with respect to this basis.

1.18 Suppose that $t \in \mathcal{L}(V, V)$ has zero as an eigenvalue. Prove that t is not invertible. Is it true that t is invertible if and only if all the eigenvalues of t are non-zero? If t is invertible, how are the eigenvalues of t related to those of t^{-1}?

1.19 Let V be a vector space of finite dimension over $\mathbb{Q}$ and let $t \in \mathcal{L}(V, V)$ be such that $t^m = 0$ for some $m > 0$. Prove that all the eigenvalues of t are zero. Deduce that if $t \neq 0$ then t is not diagonalisable.

1.20 Consider $t \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ given by

$$t(a, b) = (a + 4b, \tfrac{1}{2}a - b).$$

Find the minimum polynomial of t.

1.21 Let F be a field and let $F_{n+1}[X]$ be the vector space of polynomials of degree less than or equal to n over F. Define $t : F_{n+1}[X] \to F_{n+1}[X]$ by $t(f(X)) = f(X + 1)$. Show that t is linear.

Find the matrix of t relative to the basis $\{1, X, \ldots, X^n\}$ of $F_{n+1}[X]$. Find also the eigenvalues of t. If $g(X) = (X - 1)^{n+1}$ show that $g(t) = 0$. Hence find the minimum polynomial of t.

1.22 If V is a finite-dimensional vector space and $t \in \mathcal{L}(V, V)$ is such that $t^2 = \mathrm{id}_V$ prove that the sum of the eigenvalues of t is an integer.

1.23 For each of the following real matrices, determine

(i) the eigenvalues;
(ii) the geometric multiplicity of each eigenvalue;
(iii) whether it is diagonalisable.

For those matrices A that are diagonalisable, find an invertible matrix P such that $P^{-1}AP$ is diagonal.

$$(a)\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix},\quad (b)\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix},\quad (c)\begin{bmatrix} 7 & -1 & 2 \\ -1 & 7 & 2 \\ -2 & 2 & 10 \end{bmatrix},$$

$$(d)\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix},\quad (e)\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ -1 & 0 & 3 \end{bmatrix}.$$

1.24 Consider the sequence described by

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \ldots, \frac{a_n}{b_n}, \ldots$$

where $a_{n+1} = a_n + 2b_n$ and $b_{n+1} = a_n + b_n$.

Find a matrix A such that

$$\begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix} = A \begin{bmatrix} a_n \\ b_n \end{bmatrix}.$$

By diagonalising A, obtain explicit formulae for a_n and b_n and hence show that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \sqrt{2}.$$

1.25 Let t be a singular transformation on a real vector space V. Let $f(X)$ and $g(X)$ be real polynomials whose highest common factor is 1. Let $a = f(t)$ and $b = g(t)$.

Prove that every eigenvector of $a \circ b$ that is associated with the eigenvalue 0 is the sum of an eigenvector of a associated with the eigenvalue 0 and an eigenvector of b associated with the eigenvalue 0.

1.26 Suppose that $s, t \in \mathcal{L}(V, V)$ each have $n = \dim V$ distinct eigenvalues and that $s \circ t = t \circ s$. Prove that, for every λ in the ground field F,

$$C_\lambda = \{v \in V \mid t(v) = \lambda v\}$$

is a subspace of V. Show that C_λ is s–invariant and that, when λ_i is an eigenvalue of t, the subspace C_{λ_i} has dimension 1.

Hence show that the matrix of s with respect to the basis of eigenvectors of t is diagonal.