Algebra through practice

A collection of problems in algebra with solutions

Book 2
Matrices and vector spaces

T.S.BLYTH ○ E.F.ROBERTSON
University of St Andrews

The right of the University of Cambridge to print and sell all manner of books was granted by Henry VIII in 1534. The University has printed and published continuously since 1584.

CAMBRIDGE UNIVERSITY PRESS
Cambridge
London New York New Rochelle
Melbourne Sydney

Published by the Press Syndicate of the University of Cambridge
The Pitt Building, Trumpington Street Cambridge CB2 1RP
32 East 57th Street, New York, NY 10022, USA
296 Beaconsfield Parade, Middle Park, Melbourne 3206, Australia

© Cambridge University Press 1984

First published 1984

Library of Congress catalogue card number: 83-24013

British Library cataloguing in publication data

Blyth, T. S.
Algebra through practice.
Book 2: Matrices and vector spaces

1. Algebra – Problems, exercises, etc.
I. Title II. Robertson, E. F.
512'.0076 QA157

ISBN 0 521 27286 6

Transferred to digital printing 2003

DJ

Contents

1: Matrices and linear equations

The basic laws of matrix algebra: associativity of products (when they are defined), distributivity of multiplication over addition (when meaningful), and the like should be well known to the reader. We often use notation such as $\mathrm{Mat}_{m\times n}(\mathbb{R})$ to denote the set of $m \times n$ matrices whose entries belong to the set $\mathbb{R}$ of real numbers. We often write $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$ in the form $A = [a_{ij}]_{m\times n}$, and the identity $n \times n$ matrix as $I_n = [\delta_{ij}]$ where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$$

We assume that the reader is familiar with the transpose $A^t = [a_{ji}]_{n\times m}$ of $A = [a_{ij}]_{m\times n}$, with the notions of symmetric ($A^t = A$) and skew-symmetric ($A^t = -A$) matrices and the properties $(A^t)^t = A$, $(A + B)^t = A^t + B^t$, $(AB)^t = B^tA^t$.

We have included in this introductory section a few questions relating matrices to two-dimensional coordinate geometry, which is one of the applications of matrices that is usually covered at this level.

Another basic application is to the solution of systems of linear equations. We assume that the reader has the necessary background knowledge. This includes the reduction of an $m \times n$ matrix to row-echelon form and then to Hermite (normal) form. In particular, the reader should know that the rank of an $m \times n$ matrix is the number of non-zero rows in any row-echelon form, and that a system of equations $A\boldsymbol{x} = \boldsymbol{b}$ has a solution if and only if the rank of the coefficient matrix A is the same as that of the augmented matrix $A\,|\,\boldsymbol{b}$.

1.1 Compute the following matrix products:

$$\begin{bmatrix}1 & 3 & 0 & 4\\ 2 & 1 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\\ 1 & 0\\ 0 & 1\end{bmatrix}; \quad \begin{bmatrix}1 & 3 & 2\\ 2 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & 1\\ 1 & 1\end{bmatrix};$$

$$\begin{bmatrix}3 & 1 & -2\\ 2 & -2 & 0\\ -1 & 1 & 2\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 1 & -1 & 1\\ 0 & 1 & 2\end{bmatrix}; \quad \begin{bmatrix}1\\ 2\\ 3\\ 4\end{bmatrix}\begin{bmatrix}1 & 2 & 3 & 4\end{bmatrix}$$

$$\begin{bmatrix}1 & 2 & 3 & 4\end{bmatrix}\begin{bmatrix}1\\ 2\\ 3\\ 4\end{bmatrix}.$$

1.2 Compute the matrix product

$$\begin{bmatrix}x & y & 1\end{bmatrix}\begin{bmatrix}a & h & g\\ h & b & f\\ g & f & c\end{bmatrix}\begin{bmatrix}x\\ y\\ 1\end{bmatrix}.$$

Hence express in matrix notation the equations

(*a*) $x^2 + 9xy + y^2 + 8x + 5y + 2 = 0$;

(*b*) $\dfrac{x^2}{\alpha^2} + \dfrac{y^2}{\beta^2} = 1$;

(*c*) $xy = \alpha^2$;

(*d*) $y^2 = 4\alpha x$.

1.3 If $A = \begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}$ and $B = \begin{bmatrix}-1 & -1\\ 0 & 0\end{bmatrix}$ prove that

$$(A + B)^2 \neq A^2 + 2AB + B^2$$

but that

$$(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3.$$

1.4 Let A be the matrix

$$\begin{bmatrix} 0 & a & a^2 & a^3 \\ 0 & 0 & a & a^2 \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Define the matrix B by

$$B = A - \tfrac{1}{2}A^2 + \tfrac{1}{3}A^3 - \tfrac{1}{4}A^4 + \cdots$$

Show that this series has only finitely many terms different from zero and calculate B. Show also that the series

$$B + \frac{1}{2!}B^2 + \frac{1}{3!}B^3 + \cdots$$

has only a finite number of non-zero terms and that its sum is A.

1.5 Find all $X \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$ such that $X^2 = I_2$.

1.6 Find all $A \in \mathrm{Mat}_{2\times 2}(\mathbb{C})$ such that $A^2 = -I_2$. Show that there are no 2×2 real diagonal matrices A with $A^2 = -I_2$, but that there are infinitely many 2×2 real matrices A with $A^2 = -I_2$. Deduce that for every even positive integer n there are infinitely many $n \times n$ real matrices A with $A^2 = -I_n$. Is this so for every odd positive integer n?

1.7 Show that every $A \in \mathrm{Mat}_{2\times 2}(\mathbb{C})$ which is such that $A^2 = 0$ may be written in the form

$$\begin{bmatrix} ab & a^2 \\ -b^2 & -ab \end{bmatrix}$$

for some $a, b \in \mathbb{C}$. Is it true that every $A \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$ such that $A^2 = 0$ is of this form with $a, b \in \mathbb{R}$?

1.8 Show that a real 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ may be expressed as a product $\begin{bmatrix} x \\ y \end{bmatrix} [u \quad v]$ for some $x, y, u, v \in \mathbb{R}$ if, and only if, $ad = bc$.

1.9 For $A, B \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ define $[AB] = AB - BA$.

(*a*) Prove that the following identities hold:

(i) $[[AB]C] + [[BC]A] + [[CA]B] = 0$;

(ii) $[(A + B)C] = [AC] + [BC]$;

(iii) $[[[AB]C]D] + [[[BC]D]A] + [[[CD]A]B] + [[[DA]B]C] = 0$.

(*b*) Show by means of an example that in general

$$[[AB]C] \neq [A[BC]].$$

1.10 Consider the complex 2×2 matrices

$$X = \begin{bmatrix} \frac{1}{2}i & 0 \\ 0 & -\frac{1}{2}i \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & \frac{1}{2}i \\ \frac{1}{2}i & 0 \end{bmatrix}.$$

(*a*) Show that $AB = -BA$ for all $A, B \in X, Y, Z$ with $A \neq B$.

(*b*) Compute $AB - BA$ for each distinct pair $A, B \in \{X, Y, Z\}$ and comment on your answer.

(*c*) Prove that the 3×3 real matrices

$$X' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad Y' = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad Z' = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

satisfy the same properties as X, Y, Z.

1.11 (*a*) Show that if A and B are 2×2 matrices then the sum of the diagonal elements of $AB - BA$ is zero.

(*b*) If E is a 2×2 matrix and the sum of the diagonal elements of E is zero show that $E^2 = \lambda I_2$ for some scalar λ.

(*c*) Deduce from (*a*) and (*b*) that if A, B, C are 2×2 matrices then

$$(AB - BA)^2 C = C(AB - BA)^2.$$

1.12 Let A, B be $n \times n$ matrices with A symmetric and B skew-symmetric. Determine which of the following are symmetric and which are skew-symmetric:

$$AB + BA; \quad AB - BA; \quad A^2; \quad B^2; \quad A^p B^q A^p.$$

1.13 Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be $n \times 1$ matrices. Show that the matrix $A = \boldsymbol{x}\boldsymbol{y}^t - \boldsymbol{y}\boldsymbol{x}^t$ is of size $n \times n$ and is skew-symmetric. Show also that $\boldsymbol{x}^t\boldsymbol{y}$ and $\boldsymbol{y}^t\boldsymbol{x}$ are of size 1×1 and are equal.

If $\boldsymbol{x}^t\boldsymbol{x} = \boldsymbol{y}^t\boldsymbol{y} = [1]$ and $\boldsymbol{x}^t\boldsymbol{y} = \boldsymbol{y}^t\boldsymbol{x} = [k]$, prove that $A^3 = (k^2 - 1)A$.

1.14 If A is a square matrix such that $A^2 = A$ and $(A - A^t)^2 = 0$ prove that $(AA^t)^2 = AA^t$.

1.15 Suppose that in the cartesian plane the coordinate axes are rotated in an anti-clockwise direction through an angle ϑ. Show that the 'new' coordinates (x', y') of the point P whose 'old' coordinates are (x, y) are given by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = R_\vartheta \begin{bmatrix} x \\ y \end{bmatrix}$$

where R_ϑ is the rotation matrix

$$\begin{bmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{bmatrix}.$$

Prove that, for rotations ϑ and φ,

$$R_\vartheta R_\varphi = R_{\vartheta+\varphi} = R_\varphi R_\vartheta.$$

The hyperbola $x^2 - y^2 = 1$ is rotated anti-clockwise about the origin through 45°; find its new equation.

1.16 Two similar sheets of graph paper are pinned together at the origin and the sheets are rotated. If the point $(1, 0)$ of the top sheet lies directly above the point $(\frac{5}{13}, \frac{12}{13})$ of the bottom sheet, above what point of the bottom sheet does the point $(2, 3)$ of the top sheet lie?

1.17 For every point (x, y) of the cartesian plane let (x', y') be its reflection in the x-axis. Find the matrix M such that

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = M \begin{bmatrix} x \\ y \end{bmatrix}.$$

1.18 In the cartesian plane let L be a line passing through the origin and making an angle ϑ with the x-axis. For every point (x, y) of the plane let (x_L, y_L) be its reflection in the line L. Prove that

$$\begin{bmatrix} x_L \\ y_L \end{bmatrix} = \begin{bmatrix} \cos 2\vartheta & \sin 2\vartheta \\ \sin 2\vartheta & -\cos 2\vartheta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

1.19 In the cartesian plane let L be a line passing through the origin and making an angle ϑ with the x-axis. For every point (x, y) of the plane let (x^*, y^*) be the foot of the perpendicular from (x, y) onto L. Prove that

$$\begin{bmatrix} x^* \\ y^* \end{bmatrix} = \begin{bmatrix} \cos^2\vartheta & \sin\vartheta\cos\vartheta \\ \sin\vartheta\cos\vartheta & \sin^2\vartheta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

1.20 Find the Hermite normal form of each of the following matrices:

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 5 & 5 & 8 \end{bmatrix}; \quad \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}; \quad \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix}.$$

1.21 Reduce to row-echelon form the augmented matrix of the system of equations

$$\begin{aligned} x + 2y \qquad + 3t &= 1 \\ x + 2y + 3z + 3t &= 3 \\ x \qquad + z + t &= 3 \\ x + y + z + 2t &= 1. \end{aligned}$$

Deduce that the system has no solution.

1.22 For what value of λ does the system of equations

$$\begin{aligned} x + y \qquad + t &= 4 \\ 2x \qquad\qquad - 4t &= 7 \\ x + y + z \qquad &= 5 \\ x - 3y - z - 10t &= \lambda \end{aligned}$$

have a solution? Find the general solution when λ takes this value.

1.23 What conditions must the integers a, b, c satisfy in order that the system of equations

$$\begin{aligned} 2w - x + y - 3z &= a \\ w + x - y \qquad &= b \\ 4w + x - y - 3z &= c \end{aligned}$$

has integer solutions?

1.24 If $a, b, c, d \in \mathbb{R}$ are all greater than 0 prove that the system of equations

$$\begin{aligned} x + y + z + t &= a \\ x - y - z + t &= b \\ -x - y + z + t &= c \\ -3x + y - 3z - 7t &= d \end{aligned}$$

has no solutions.

1.25 Show that the equations

$$\begin{aligned} 2x + y + \qquad z &= -6\beta \\ \gamma x + 3y + \qquad 2z &= 2\beta \\ 2x + y + (\gamma + 1)z &= 4 \end{aligned}$$

have a unique solution except when $\gamma = 0$ and when $\gamma = 6$. If $\gamma = 0$ prove that there is only one value of β for which a solution exists and find the general solution in this case. Discuss the situation when $\gamma = 6$.

1.26 Show that the equations

$$\begin{aligned} x - y \qquad\quad - u - 5t &= \alpha \\ 2x + y - z - 4u + t &= \beta \\ x + y + z - 4u - 6t &= \gamma \\ x + 4y + 2z - 8u - 5t &= \delta \end{aligned}$$

have a solution if and only if

$$8\alpha - \beta - 11\gamma + 5\delta = 0.$$

Find the general solution when $\alpha = \beta = -1, \gamma = 3, \delta = 8$