Cambridge University Press
978-0-521-27285-8 - Algebra Through Practice: A Collection of Problems in Algebra with Solutions, Sets, Relations and Mappings
T. S. Blyth and E. F. Robertson
Excerpt
More information

1: Sets

We assume that the reader has a basic knowledge of elementary set theory and we shall use standard (i.e. the most commonly accepted) notation. Thus, for example, we shall denote the complement of a subset A of a set E simply by A' except when confusion can occur in which case we shall write $C_E(A)$. If A and B are subsets of E then the difference set $A \cap B'$ will be denoted by $A \setminus B$ (some authors use $A - B$), and the symmetric difference set $(A \cap B') \cup (A' \cap B)$ will be denoted by $A \,\Delta\, B$.

Some questions in this section are best dealt with using the algebra of set theory, with which we assume that the reader is familiar. For example, this includes the distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

and

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$$

and the de Morgan laws

$$(A \cap B)' = A' \cup B' \quad \text{and} \quad (A \cup B)' = A' \cap B'.$$

Other questions, particularly those dealing with set-theoretic identities, are best dealt with using Venn diagrams.

Other standard notation that we shall employ includes $\mathbf{P}(E)$ for the power set of E (i.e. the set of all subsets of E); $|A|$ for the number of elements in the set A; $A \times B$ for the cartesian product of A and B (i.e. the set of ordered pairs (a, b) with $a \in A$ and $b \in B$); and the following for particular subsets of the number system:

$\mathbb{N} = \{0, 1, 2, \ldots\}$ for the set of natural numbers;

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ for the set of integers;

$\mathbb{Q} = \{a/b \mid a, b \in \mathbb{Z}, b \neq 0\}$ for the set of rationals;

$\mathbb{R}$ for the set of real numbers;

$\mathbb{C}$ for the set of complex numbers;

$]a,b] = \{x \in \mathbb{R} \mid a < x \leqslant b\}$;

$]a,b[= \{x \in \mathbb{R} \mid a < x < b\}$, etc.

Finally the usual logical abbreviations $\exists$ (there exists), $\forall$ (for all), $\Rightarrow$ (implies), $\Leftrightarrow$ (if and only if) will be used throughout, both in the problems and in their solutions.

1.1 Let $A = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$. Which of the following are true?

$$\emptyset \subseteq A,\ \emptyset \in A,\ \{\emptyset\} \in A,\ \{\emptyset\} \subseteq A,\ \{\{\emptyset\}\} \subseteq A,\ \{\{\emptyset\}, \emptyset\} \subseteq A,$$
$$\{\{\emptyset\}, \emptyset\} \in A.$$

1.2 List the elements of $\mathbf{P}(\mathbf{P}(\emptyset))$ and of $\mathbf{P}(\mathbf{P}(\mathbf{P}(\emptyset)))$.

1.3 For the set $E = \{1, \{1\}, 2, \{1, 2\}\}$ determine $\mathbf{P}(E)$ and $E \cap \mathbf{P}(E)$.

1.4 Find four examples of a set A with the property that every element of A is a subset of A.

1.5 Can you find sets A, B, C such that

$$A \subseteq B \in C \quad \text{and} \quad A \in B \subseteq C?$$

1.6 Which of the following hold for all sets A, B and C?

(*a*) If $A \notin B$ and $B \notin C$ then $A \notin C$.
(*b*) If $A \neq B$ and $B \neq C$ then $A \neq C$.
(*c*) If $A \in B$ and $B \not\subseteq C$ then $A \notin C$.
(*d*) If $A \subseteq B$ and $B \subseteq C$ then $C \not\subseteq A$.
(*e*) If $A \subseteq B$ and $B \in C$ then $A \notin C$.
(*f*) If $A \cap C \subseteq B$ then $(A \cap B) \cup (B \cap C) = B$.
(*g*) If $A \cap C \in B$ then $A \in B \cup C$.
(*h*) If $A \cap C = \emptyset$ and $B \cap C = \emptyset$ then $(A \cup B) \cap C = \emptyset$.

1.7 Show that $\{x, y\} \cap \{y, z\} = \{y\}$ may be false.

1.8 Let A, B, C be subsets of a set X. Simplify the expressions

(*a*) $(A \cup (A \cup B)')'$;
(*b*) $((A \cup \emptyset) \cap (B \cup A') \cap (A \cup B' \cup X))'$;
(*c*) $(A \cup (B \cap C) \cup (B' \cap C') \cup C)'$.

1.9 Let A, B be subsets of a set X. Prove, using a Venn diagram, that

$$(A \cap B') \cup (A' \cap B) = A \cup B \Leftrightarrow A \cap B = \emptyset.$$

1.10 Let A, B, C be subsets of a set X. Prove, using Venn diagrams, that

(*a*) $A \mathbin{\Delta} (B \mathbin{\Delta} C) = (A \mathbin{\Delta} B) \mathbin{\Delta} C$;

(*b*) $A \cup B = A \mathbin{\Delta} B \mathbin{\Delta} (A \cap B)$;

(*c*) $A \cap (B \mathbin{\Delta} C) = (A \cap B) \mathbin{\Delta} (A \cap C)$;

(*d*) $A \mathbin{\Delta} (B \cap C) = (A \mathbin{\Delta} B) \cap (A \mathbin{\Delta} C)$ if and only if $A \cap B = A \cap C$.

1.11 If A, B are subsets of a set E let $A \mid B = C_E(A \cap B)$. Show that $C_E(A) = A \mid A$ and that $A \cap B = (A \mid B) \mid (A \mid B)$. Express $\cup$ in terms of $\mid$ alone.

1.12 Let A, B, C, D be sets with $\{A, B\} = \{C, D\}$. Prove that $A \cap B = C \cap D$ and that $A \cup B = C \cup D$.

1.13 If A, B, C are sets prove that $A \cap (B \cup C) \subseteq (A \cap B) \cup C$, with equality if and only if $C \subseteq A$.

1.14 For sets E, F, G prove, using a Venn diagram, that

$$(E \cap F) \cup (F \cap G) \cup (G \cap E) = (E \cup F) \cap (F \cup G) \cap (G \cup E).$$

1.15 Give an example of sets A, B, C, D with

$$(A \cup B) \times (C \cup D) \neq (A \times C) \cup (B \times D).$$

1.16 Given two objects x, y one may define the ordered pair (x, y) by

$$(x, y) = \{\{x\}, \{x, y\}\}.$$

Use this definition to prove that $(x, y) = (x^*, y^*)$ if and only if $x = x^*$ and $y = y^*$. Prove also that, for every object x,

$$\{x\} \times \{x\} = \{\{\{x\}\}\}.$$

1.17 Prove that if A, B, C are sets with A and B not empty then

$$(A \times B) \cup (B \times A) = C \times C \Leftrightarrow A = B = C.$$

1.18 Let $\mathscr{F}$ be a collection of sets such that

$$X, Y \in \mathscr{F} \Rightarrow X \setminus Y \in \mathscr{F}.$$

Prove that if $X, Y \in \mathscr{F}$ then $X \cap Y \in \mathscr{F}$.

1.19 Let $\mathscr{F}$ be a collection of sets. Define

$$\mathscr{F}^0 = \{X \setminus Y \mid X, Y \in \mathscr{F}\}.$$

Show that $\mathscr{F}^0 \subseteq (\mathscr{F}^0)^0$. Give an example to show that it is possible to have $\mathscr{F}^0 \neq (\mathscr{F}^0)^0$.

1.20 Let A, B be sets. Are the following true?

(*a*) $\mathbf{P}(A) \cap \mathbf{P}(B) = \mathbf{P}(A \cap B)$;

(*b*) $\mathbf{P}(A) \cup \mathbf{P}(B) = \mathbf{P}(A \cup B)$.

1.21 If A, B, C are sets prove, using Venn diagrams, that $A \setminus (B \cup C) = (A \setminus B) \cup (A \setminus C)$ if and only if $A \mathbin{\Delta} (B \cup C) = (A \mathbin{\Delta} B) \cup (A \mathbin{\Delta} C)$. Find sets A, B, C with

$$A \mathbin{\Delta} (B \cup C) = (A \mathbin{\Delta} B) \cup (A \mathbin{\Delta} C);$$
$$A \mathbin{\Delta} (B \cup C) \neq (A \mathbin{\Delta} B) \cup (A \mathbin{\Delta} C).$$

1.22 Let $A_1, \ldots, A_m$ be subsets of a set E. Define $B_1, \ldots, B_m$ recursively as follows:

$$B_1 = A_1, \quad (\forall n \geqslant 2)\ B_n = A_n \setminus \bigcup_{k=1}^{n-1} A_k.$$

Show that $B_i \cap B_j = \emptyset$ for $i \neq j$ and that $\bigcup_{i=1}^{m} B_i = \bigcup_{i=1}^{m} A_i$.

1.23 Let A, B be sets with $A \subseteq B$. Prove that there is a unique subset X of B such that $X \cup A = B$ and $X \cap A = \emptyset$.

1.24 An examination in three subjects algebra (A), biology (B), chemistry (C) was taken by 41 students. The following table shows how many students failed the various combinations of subjects.

Subjects	A	B	C	A, B	A, C	B, C	A, B, C
No. of failed students	12	5	8	2	6	3	1

How many students passed all three subjects?

1.25 At least 70% of a class of students study algebra, at least 75% study calculus, at least 80% study geometry, and at least 85% study trigonometry. What percentage (at least) must study all four subjects?

1.26 Let E be a set consisting of n elements. If X, Y are subsets of E such that the number of elements in the sets $X \cap Y, X' \cap Y, X \cap Y', X' \cap Y'$ are p, q, r, s respectively, prove that $p + q + r + s = n$.

In a sixth form of n girls and boys each pupil is either an arts student or a science student. If the proportion of arts students among the girls is greater than the proportion of arts students among the boys, show that the proportion of girls among the arts students is greater than the proportion of girls among the science students.

1.27 If $A = \{x \in \mathbb{Z} \mid (\exists y \in \mathbb{Z})x = 2y\}$ and $B = \{a \in \mathbb{Z} \mid (\exists b, c \in \mathbb{Z})a = 6b + 10c\}$ prove that $A = B$.

1.28 For which $S \in \{\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}\}$ are the following statements true?

(*a*) $\{x \in S \mid x^2 = 5\} \neq \emptyset$;

(*b*) $\{x \in S \mid |x-1| \leqslant \frac{1}{2}\} = \{1\}$;
(*c*) $\{x \in S \mid x^2 = -1\} = \emptyset$.

1.29 Given $n \in \mathbb{N}$ define $n\mathbb{Z} = \{nx \mid x \in \mathbb{Z}\}$. Is it true that given $n_1, n_2 \in \mathbb{N}$ there exists $m \in \mathbb{N}$ with

$$n_1\mathbb{Z} \cap n_2\mathbb{Z} = m\mathbb{Z}?$$

Is it true that, for some $m \in \mathbb{N}$,

$$n_1\mathbb{Z} \cup n_2\mathbb{Z} = m\mathbb{Z}?$$

1.30 Let $A = \{x \in \mathbb{N} \mid 1 \leqslant x \leqslant n\}$.

(*a*) How many subsets does A have?
(*b*) How many subsets of A contain at least one even integer?
(*c*) How many subsets of A contain exactly one even integer?

(*Hint:* consider separately the cases n even, n odd.)

1.31 Let $A = \{x \in \mathbb{N} \mid 1 \leqslant x \leqslant n\}$. What is the maximum possible k for which $A_i \subseteq A$ $(i = 1, \ldots, k)$ and $A_i \subset A_j$ if $i < j$? Find $\sum_{i=1}^{k} |A_i|$.

1.32 Express as a union of intervals

$$\left\{x \in \mathbb{R} \setminus \{-1, 4\} \;\middle|\; \frac{1}{(x+1)(x-4)} > -\frac{1}{4}\right\}.$$

1.33 Express as a union of intervals the set of real numbers k for which

$$\left\{x \in \mathbb{R} \;\middle|\; \frac{(x-1)^2}{(x+1)(x+3)} = k\right\} = \emptyset.$$

Cambridge University Press
978-0-521-27285-8 - Algebra Through Practice: A Collection of Problems in Algebra with Solutions, Sets, Relations and Mappings
T. S. Blyth and E. F. Robertson
Excerpt
More information

2: Relations

A relation R between a set E and a set F is a subset of $E \times F$, and we shall use the notation $(x, y) \in R$ or xRy with the same meaning. When we think of a coordinate pictorial representation of $E \times F$ we refer to $\{(x, y) \mid xRy\}$ as the graph of R, to $\{x \in E \mid (\exists y \in F)(x, y) \in R\}$ as the domain of R, and to $\{y \in F \mid (\exists x \in E)(x, y) \in R\}$ as the image of R. A relation between E and E is called a (binary) relation on E.

If the relation R on E is reflexive (xRx for all $x \in E$), symmetric (if xRy then yRx), and transitive (if xRy and yRz then xRz), then R is an equivalence relation on E. When R is an equivalence relation on E we sometimes write $x \equiv y(R)$ instead of xRy. For $x \in E$ the R-class of x, i.e. $\{y \in E \mid yRx\}$, is denoted by $[x]_R$ or simply $[x]$ when no confusion can arise. The following are equivalent:

$$x \equiv y(R), \quad y \in [x]_R, \quad [x]_R = [y]_R, \quad [x]_R \cap [y]_R \neq \emptyset.$$

It follows that two R-classes either are disjoint (i.e. have empty intersection) or are identical. This leads to the notion of a partition of E as a collection of non-empty subsets of E which are pairwise disjoint and whose union is the whole of E. If R is an equivalence relation on E then the R-classes form a partition of E. Conversely, every partition of E defines an equivalence relation $\equiv$ on E by

$$x \equiv y \Leftrightarrow x, y \text{ belong to the same subset in the partition.}$$

An example of an equivalence relation is the relation mod n defined on $\mathbb{Z}$ by $a \equiv b(\text{mod } n)$ if and only if n divides $a - b$. The corresponding partition consists of the equivalence classes

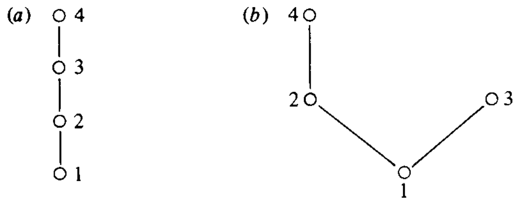
$$[0] = \{\ldots, -2n, -n, 0, n, 2n, \ldots\}$$
$$[1] = \{\ldots, -n+1, 1, n+1, \ldots\}$$
$$\vdots$$
$$[n-1] = \{\ldots, -1, n-1, 2n-1, \ldots\}$$

A relation R on a set E that is reflexive, anti-symmetric (if xRy and yRx then $x = y$), and transitive is called an order (or a partial order) and is often written $\leqslant$. When $a \leqslant b$ and $a \neq b$ we write $a < b$. Examples of order relations are

(*a*) $\subseteq$ on $\mathbf{P}(E)$;
(*b*) $|$ on $\mathbb{Z}$, where $a \mid b \Leftrightarrow a$ divides b.

An order relation can often be represented pictorially by a Hasse diagram. In this, $a < b$ is exhibited by joining the point representing a to the point representing b by an increasing line segment. For example, the Hasse diagrams for the set $\{1, 2, 3, 4\}$ ordered first in the usual way and then by divisibility are shown in Figs 2.1 (*a*) and (*b*) respectively.

Fig.2.1



2.1 Let S be the relation defined on $\mathbb{R}$ by
$$xSy \Leftrightarrow x^2 = x|y+1|.$$
Sketch the graph of S.

2.2 Sketch the graphs of the following relations on $\mathbb{R}$:
(*a*) $\{(x,y) \mid |x+y| \leqslant 1\}$;
(*b*) $\{(x,y) \mid 2x^2 + 3xy - 2y^2 \leqslant 0\}$;
(*c*) $\{(x,y) \mid (x-y)(x-2y)(x-3y) \geqslant 0\}$;
(*d*) $\{(x,y) \mid x+y-4 \leqslant 0,\ 2x-y-4 \leqslant 0,\ 2x-5y-10 \leqslant 0,$ $3x-y+3 \geqslant 0\}$.

2.3 Let $A = \{1, 2, 3, 4\}$. Determine the graphs of the relations R, S defined on A by

$$aRb \Leftrightarrow a + b \leqslant 4;$$
$$aSb \Leftrightarrow a(b+1) \leqslant 6.$$

2.4 Relations R_1 and R_2 are defined on $\mathbb{R}$ by

$$xR_1y \Leftrightarrow -10 \leqslant x + 5y \leqslant 10,$$
$$xR_2y \Leftrightarrow x^2 + y^2 \leqslant 4, x \geqslant y.$$

Sketch the graphs of these relations.

2.5 Let the relation ρ on a set A have the properties

(*a*) $a\rho a$ for every $a \in A$;

(*b*) if $a\rho b$ and $b\rho c$ then $c\rho a$.

Prove that ρ is an equivalence relation on A. Does every equivalence relation on A satisfy (*a*) and (*b*)?

2.6 Consider the relation $R = \{(a, b), (a, c), (a, a), (b, d), (c, c)\}$ defined on the set $X = \{a, b, c, d\}$. Find the minimum number of elements of $X \times X$ which must be adjoined to R in order to make R

(*a*) reflexive;

(*b*) symmetric;

(*c*) an equivalence relation.

Answer the same questions for $S = \{(a, b), (a, c), (a, a), (c, c)\}$.

2.7 How many different equivalence relations can be defined on the set $\{a, b, c\}$?

2.8 Given relations R, S on a set A, define the product relation RS by

$$(x, y) \in RS \Leftrightarrow (\exists z \in A)((x, z) \in S \text{ and } (z, y) \in R).$$

Give an example of relations R, S with $RS = SR$ and an example of relations R, S with $RS \neq SR$.

Prove that if R and S are equivalence relations then RS is an equivalence relation if and only if $RS = SR$. Deduce that RS is an equivalence relation if and only if SR is an equivalence relation.

2.9 Let R_1, R_2 and S be relations on a set X. Prove that

(*a*) if $R_1 \subseteq R_2$ then $SR_1 \subseteq SR_2$ and $R_1S \subseteq R_2S$;

(*b*) $S(R_1 \cup R_2) = SR_1 \cup SR_2$.

2.10 If R_1 and R_2 are equivalence relations on a set X prove that

(*a*) $R_1 \cap R_2$ is an equivalence relation;

(*b*) $R_1 \cup R_2$ need not be an equivalence relation.

Give an example of equivalence relations R_1 and R_2 with $R_1 \neq R_2$ and $R_1 \cup R_2$ an equivalence relation.

2.11 Let α be the relation on $\mathbb{N}$ defined by

$$a\alpha b \Leftrightarrow a^2 \equiv b^2 (\text{mod } 7).$$

Show that α is an equivalence relation. Into how many equivalence classes does α partition $\mathbb{N}$?

2.12 Let $S = \mathbb{R} \setminus \{0\}$. Define a relation ρ on $S \times S$ by

$$(a, b)\rho(c, d) \Leftrightarrow c^2 b = a^2 d.$$

Prove that ρ is an equivalence relation. Describe geometrically the ρ-classes.

If σ is defined on $S \times S$ by

$$(a, b)\sigma(c, d) \Leftrightarrow c^4 b^2 = a^4 d^2,$$

show that σ is an equivalence relation. Describe geometrically the σ-classes. Explain how the equivalence classes of ρ and of σ are related.

If the relation τ is defined on $\mathbb{R} \times \mathbb{R}$ by

$$(a, b)\tau(c, d) \Leftrightarrow c^2 b = a^2 d,$$

is τ an equivalence relation?

2.13 Consider the relation $\sim$ defined on $\mathbb{C} \setminus \{0\}$ by

$$z_1 \sim z_2 \Leftrightarrow |z_1|(|z_2|^2 + 1) = |z_2|(|z_1|^2 + 1).$$

Prove that $\sim$ is an equivalence relation. If $a \in \mathbb{R}$ is such that $0 < a < 1$, sketch on the Argand diagram the $\sim$-class of a.

2.14 Consider the relation $\sim$ defined on $\mathbb{C} \setminus \{0\}$ by

$$z_1 \sim z_2 \Leftrightarrow z_1 \bar{z}_1 (z_2 + \bar{z}_2) = z_2 \bar{z}_2 (z_1 + \bar{z}_1).$$

Prove that $\sim$ is an equivalence relation. If a is a non-zero number on the real axis, give a geometrical description of the $\sim$-class of a.

2.15 Let $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x \neq 0, y \neq 0\}$ and define a relation $\sim$ on S by

$$(x_1, y_1) \sim (x_2, y_2) \Leftrightarrow x_1 y_1 (x_2^2 - y_2^2) = x_2 y_2 (x_1^2 - y_1^2).$$

(*a*) Show that $\sim$ is an equivalence relation.

(*b*) If (a, b) is a fixed element of S show that

$$(x, y) \sim (a, b) \Leftrightarrow \frac{y}{x} = \frac{b}{a} \quad \text{or} \quad \frac{y}{x} = -\frac{a}{b}.$$

(*c*) Sketch the $\sim$-class containing (2,1).

2.16 If a is a given real number consider the relation f on $\mathbb{R}$ given by

$$(x, y) \in f \Leftrightarrow y = x^2 + ax + a^2.$$

(*a*) By considering the graph of f prove that if $A = \{x \in \mathbb{R} \mid (x, 1) \in f\}$ then

(i) $A = \emptyset$ if and only if $|a| > 2/\sqrt{3}$;
(ii) $|A| = 1$ if and only if $|a| = 2/\sqrt{3}$;
(iii) $|A| = 2$ if and only if $|a| < 2/\sqrt{3}$.

(*b*) Prove that the relation S defined on $\mathbb{R}$ by

$$x \equiv y(S) \Leftrightarrow x^3 - y^3 = x - y$$

is an equivalence relation. Deduce from the above that the S-class of $x \in \mathbb{R}$ consists of

(i) a single element if and only if $|x| > 2/\sqrt{3}$;
(ii) two elements if and only if $|x| = 2/\sqrt{3}$ or $|x| = 1/\sqrt{3}$;
(iii) three elements otherwise.

2.17 (*a*) Prove that the relation R defined on $\mathbb{R}$ by

$$xRy \Leftrightarrow x^2 - y^2 = 2(y - x)$$

is an equivalence relation. Determine the R-class of 0 and the R-class of 1.

(*b*) The following argument leads to a false conclusion. Explain where it is incorrect.

Since $x^2 - y^2 = (x+y)(x-y)$ it follows that if $x^2 - y^2 = 2(y-x)$ then $(x+y)(x-y) = -2(x-y)$ and so $x + y = -2$. Hence the relation S defined by $xSy \Leftrightarrow x + y = -2$ is also an equivalence relation, and from $1S1$ we have $2 = -2$.

2.18 Let M be a set of $mn+1$ positive integers. Let $\equiv$ be the relation on M defined by

$$a \equiv b \Leftrightarrow a \mid b,$$

and let S be the relation on M defined by

$$aSb \Leftrightarrow a \not\equiv b \quad \text{and} \quad b \not\equiv a.$$

Show that M contains either a subset $\{a_1, a_2, \ldots, a_{m+1}\}$ with $a_i \equiv a_{i+1}$ for $1 \leqslant i \leqslant m$ or a subset $\{b_1, b_2, \ldots, b_{n+1}\}$ with $b_j S b_k$ for $j \neq k$. (*Note:* this question is quite hard.)

2.19 Let $A_1, A_2, \ldots, A_n$ be subsets of a set X. For each A_i, let A_i^0 denote A_i and let A_i^1 denote the complement of A_i in X. A *constituent* of X with respect to $A_1, \ldots, A_n$ is defined to be a non-empty subset of the form

$$A_1^{\epsilon_1} \cap A_2^{\epsilon_2} \cap \ldots \cap A_n^{\epsilon_n}$$