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PART I

THE LOCAL HOMOLOGICAL CONJECTURES, BIG COHEN-MACAULAY MODULES, AND RELATED TOPICS

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THE SYZYGY PROBLEM: A NEW PROOF AND HISTORICAL PERSPECTIVE

E.G.EVANS and PHILLIP GRIFFITH

This article is a brief survey of the results that led up to our solution of the syzygy problem [8] as well as a discussion of our solution of that problem as it was generalized during the Durham Symposium following conversations with Bruns, Foxby, Hochster, Huneke, Roberts, Szpiro and others.

From our view the syzygy problem began with three separate and unrelated events in 1969. One was the submission of Hackman's (so far unpublished) Ph.D. Thesis [11] "Exterior powers and homology", which contains on the penultimate page the statement of the problem. Using techniques of his thesis he proved that regular local rings of dimension three are unique factorization domains and writes as follows.

"In order to prove the general UFD Theorem along the same lines, one would need the following theorem: If the projective dimension of M is r and M admits a projective resolution

$$0 \to F_r \to F_{r-1} \to \dots \to F_0 \to M \to 0,$$

where the F_i, $i < r$, are finitely generated free modules and F_r is an admissible projective module, then

$$\sum_{j=k}^{r} (-1)^{j-k} \mathrm{rk}(F_j) \geq k$$

for all $k \leq r-1$; in other words, the k-th syzygy module of the resolution is of rank $\geq k$."

A second event in 1969 was the appearance of Auslander and Bridger's monograph [1] "Stable module theory" which contains among many other results an explicit criterion for a module of finite projective dimension to be a k-th syzygy. Because they were interested in a different circle of ideas and were writing in more generality than we need, it is somewhat difficult to give a concise and explicit reference to their connection with the syzygy problem.

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Perhaps the best one is their Theorem 4.25 [1; p.127] which proves that, if M has finite Gorenstein dimension, then seven statements concerning M are equivalent. We need to remark that a module of finite projective dimension has the same finite Gorenstein dimension. Two of the seven equivalent conditions are the following.

(b) There is an exact sequence $0 \to M \to P_{k-1} \to \dots \to P_1 \to P_0$ where the P_i are projective (that is, M is a k-th syzygy).

(f) For each prime ideal P every R_P regular sequence of length $\leq k$ is M_P-regular.

It is interesting to note that this memoir is an outgrowth of Bridger's Ph.D. Thesis (Brandeis).

Thirdly in 1969, Peskine and Szpiro circulated a preprint of what was to become the first two chapters of their remarkable article [17] "Dimension projective finie et cohomologie locale" which was their joint Ph.D. Thesis. This article established their famous intersection property from which they settled many open problems for rings containing a field. The intersection property (cf. [17; p.84]) is defined as follows. Let A be a local ring and M a nonzero A-module of finite projective dimension. Then A has the intersection property if for all finitely generated A-modules N, one has $\dim N_P \leq \mathrm{pd}.M$ for each prime P which is minimal in $\mathrm{Supp}\, M \cap \mathrm{Supp}\, N$. Peskine and Szpiro [17; Theorem 2.1, p.86] prove that, if A is a local ring containing a field, then all finitely generated A-modules of finite projective dimension have the intersection property. It is of some interest to note that the weaker inequality that $\mathrm{depth}\, N_P \leq \mathrm{pd}.M$, for prime ideals P minimal in $\mathrm{Supp}\, M \cap \mathrm{Supp}\, N$, is much easier to prove and, indeed, follows from their Lemme d'Acyclicité [17; p.55] which states as follows.

Let A be a local ring and let $0 \to L_s \to \dots \to L_0 \to 0$ *be a complex of finitely generated A-modules. Suppose that*

(1) $\mathrm{depth}\, L_i \geq i$ *and*

(2) $\mathrm{depth}\, H_i(L.) = 0$ *or* $H_i(L.) = 0$.

Then $H_i(L.) = 0$ *for all* $i \geq 1$.

The proof of the weaker inequality can be deduced from the lemma quite easily. First one replaces A by A_P, M by M_P and N by N_P,

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where P is a minimal prime in Supp M ∩ Supp N. This is harmless since the projective dimension of M can only get smaller. Thus P is the only prime in Supp M ∩ Supp N. Therefore all of the homology modules $\mathrm{Tor}_i^A(M,N)$ (including $M \otimes N$) have finite length. Let $0 \to L_d \to \ldots \to L_0 \to M \to 0$ be a minimal projective resolution of M. If depth N > pd.M, we apply the Lemme d'Acyclicité to the complex

$$0 \to L_d \otimes N \to \ldots \to L_0 \otimes N \to 0$$

to conclude that the complex is acyclic and, further, using a standard depth counting argument for long exact sequences, we conclude that the complex is too short to have its zero-th homology nonzero and of finite length. Thus, the real difficulty Peskine and Szpiro overcame was to be able to establish the stronger inequality one obtains by replacing depth by dimension.

It is interesting to recall that all three of these early contributions were connected with Ph.D. Theses. It is also noteworthy that the clear vision provided by hindsight has shown that the essential ingredient in the final solution came from a better understanding of the Lemme d'Acyclicité. Perhaps we should also remark that by 1970 Hartshorne [12] was developing a theme in algebraic geometry concerning questions on vector bundles of small rank and complete intersections which, as it turns out, was rather closely related to subsequent work on the syzygy problem (cf. Bruns-Evans-Griffith [4] and Hartshorne [12],[13]).

The next collection of results came in the mid 1970's. Lebelt [15], [16] tried to extend Hackman's ideas in order to settle the syzygy problem. Briefly, he wanted to compare a projective resolution of $\wedge^k M$ with that of M and then use it to show that, if M was a module of rank k which was also a "large" syzygy, then $\wedge^k M$ would necessarily be a reflexive ideal. Thus it would follow that both $\wedge^k M$ and M would be free. Lebelt managed to obtain several interesting results along these lines which provided the first explicit and affirmative results on the problem.

In 1976 Bruns [2] established that the bound given by Hackman [11] was the best possible in a very general sense. He showed that, if R is a Cohen-Macaulay domain and if M is a k-th syzygy of finite

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projective dimension of rank exceeding k, then M contains a free submodule F such that M/F is a k-th syzygy of rank exactly k. The proof consists of two parts. One part uses the basic element results of Eisenbud and Evans [5] to show that, if M has rank exceeding k, then there is an $x \in M$ such that x is a minimal generator of M_P for all prime ideals P of height k. One should note that, if k is less than dim R, then x can be taken in $\underline{m}M$, where $\underline{m}$ denotes the maximal ideal of R. The second part of the proof uses the criterion of Auslander and Bridger [1] previously mentioned to show that, if x is as described above and if M is a k-th syzygy, then M/Rx is again a k-th syzygy. Bruns' proof concludes by descending induction on the rank of M. One should note that, if M is not free, then M/F also is not free and, if M is free with $k < \dim R$, then one can take $F \subset \underline{m}M$ so that M/F has projective dimension one. In particular, non-free k-th syzygies of rank k exist in abundance as soon as non-free k-th syzygies exist. However, there is no evidence at this point to suggest that one can expect non-free k-th syzygies of rank less than k.

One may analyze the second part of Bruns' proof as follows. If M is a k-th syzygy and P is a prime ideal of height k, then M_P is a free R_P-module. If x is a minimal generator of M_P, then there is a map of M to R sending x outside P as a result of M_P being free. Thus the ideal of images of such an x, defined by

$$O_M(x) = \{f(x) \mid f \in \mathrm{Hom}(M,R)\},$$

cannot be contained inside any prime ideal of height k. Conversely, if M is a k-th syzygy and if $O_M(x)$ is an ideal of height greater than k, then M/Rx is again a k-th syzygy. This remark led Eisenbud and Evans [6] to investigate how the height of $O_M(x)$ depends on the rank of M. For if $M = R^k$ is free of rank k and $x \in \underline{m}R^k$, then $x = \langle r_1,\ldots,r_k \rangle$ with $r_i \in \underline{m}$ for each i and $O_M(x) = (r_1,\ldots,r_k)$. It follows from the Krull altitude theorem that the height of $O_M(x)$ is at most k. Eisenbud and Evans [6] established this result for arbitrary M of rank k as long as R was a domain which contained a field. These assumptions on R were later removed by Bruns [3]. Evans [7] further remarked that the preceding result showed that,

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if M was a k-th syzygy of rank k and if $x \in \underline{m}M$, $x \neq 0$, then M/Rx is never a k-th syzygy. This observation gave further evidence that the bound on rank was the correct one.

On another tack Peskine and Szpiro [18] and, independently, Roberts [20] strengthened the Lemme d'Acyclicité in that depth was replaced by dimension. More exactly they proved [18],[20] that, if R contains a field and if $0 \to L_s \to \ldots \to L_0 \to 0$ is a complex of free R-modules having finite length homology, then s is at least the dimension of R (assuming the zero-th homology is nonzero). This gave added strength to the philosophy that many inequalities using depth as a bound might remain true when dimension is used.

In 1974 Hochster [14] in his remarkable memoir "Topics in the homological theory of modules over commutative rings" validated the above philosophy. He constructed for any local ring R, which contained a field, a maximal Cohen-Macaulay module (not necessarily finitely generated). These modules, by their existence, allowed one to replace depth by dimension in many existing inequalities. In particular, the earlier result of Peskine and Szpiro [17] and Roberts [19] could be improved in this way.

Now we shall examine our original proof of the syzygy problem [8]. Our first version contained various restrictions on the ring R involved. To be precise we needed that R contains a field in order that factor rings of R have maximal Cohen-Macaulay modules (as discussed above). We needed that R is a domain so that rank is well defined and we needed the Cohen-Macaulay property in order to apply the Auslander-Bridger criterion for a module to be a k-th syzygy. During discussions at the Symposium it became clear that the last two assumptions on the ring R could be dropped. Firstly, if M has a finite free resolution, then one can define the rank of M to be the alternating sum of the Betti numbers. Secondly, if R fails to be Cohen-Macaulay, then the depth of R is smaller and it becomes even more difficult for a module to be a k-th syzygy of finite projective dimension. It also became apparent during the course of the Symposium that one could modify the proofs of Peskine and Szpiro [18] and Roberts [20] in order to provide a version of the Lemme d'Acyclicité

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suitable for one of the crucial steps in our proof. The final deviation in our new proof of the syzygy theorem (as presented below) occurs in the use of our result on order ideals of minimal generators [9]. There is a slight drawback here of a technical nature in that we established our statement concerning the heights of order ideals of minimal generators under the assumption that the residue field is algebraically closed. However we shall state a slight modification of this result which is suitable for our needs here. Except for this our proof is rather easier than our original one in addition to being more general.

Definition. Let R be a local ring and let M be an R-module having a finite free resolution $0 \to F_s \to \dots \to F_1 \to F_0 \to M \to 0$. Then the *rank of* M is defined by $\operatorname{rank} M = \sum_{i=0}^{s} (-1)^i \operatorname{rank} F_i$. Of course this definition agrees with the usual one in case R is a domain or as generalized by Bruns [3].

LEMMA. *Let* $(R,\underline{m},\underline{k})$ *be a* <u>*complete*</u> *local ring and let* M *be a finitely generated non-free* R*-module of finite projective dimension and having rank* r. *Then there is a* <u>*finite*</u> *faithfully flat residue field extension* $(R',\underline{m}',\underline{k}')$ *of* $(R,\underline{m},\underline{k})$ *and a minimal generator* x *of* $R' \otimes M$ *having order ideal of height less than or equal to* r.

Proof. The argument is essentially that given in [9]. We fix a minimal generating set $e_1,\dots,e_t$ of M, noting that $t > r$ since M is not free. Next we form the polynomial ring $S = R[X_1,\dots,X_t]$ and the S-module $N = S \otimes M'$. Considering the element $v = \sum_{i=1}^{t} X_i e_i$ of N we have from Bruns [3] that the height of the order ideal $O_N(v)$ does not exceed $r = \operatorname{rank}_R M = \operatorname{rank}_S N$. Let P be a prime ideal of height r containing $O_N(v)$. As noted in [9], the height of the ideal $P + \underline{m}S$ is at most $r + \dim R$, which is less than $\dim S$. Hence there is a maximal ideal Q (actually infinitely many) of S which contains $P + \underline{m}S$ and which corresponds modulo $\underline{m}$ to a maximal ideal of $\underline{k}[X_1,\dots,X_t]$ other than $(X_1,\dots,X_t)$. The remainder of our proof is exactly that given in [9] provided Q is of the form $Q = (\underline{m}, X_1 - a_1,\dots,X_t - a_t)$ where $a_i \in R$, the element $x = \sum_{i=1}^{t} a_i e_i$ being the desired element. However, this can be achieved after a finite extension of $\underline{k}$, since only a finite number of equations are

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involved. Since R is complete and local we may extend its residue field (finitely) so that the resulting ring $(R', \underline{m}', \underline{k}')$ is complete and local as well as being a faithfully flat extension of R. Thus we can achieve a maximal ideal Q of the desired form by passing to a suitable finite extension $(R', \underline{m}', \underline{k}')$. □

The more general solution to the syzygy problem now follows.

THEOREM. *Let* $(R, \underline{m}, \underline{k})$ *be a local ring containing a field. Let* M *be a finitely generated* k-*th syzygy of rank* r. *If* r *is less than* k, *then* M *is free.*

Proof. We may assume that M is locally free on the punctured spectrum of R since otherwise we may localize to a ring of smaller dimension while keeping M a k-th syzygy of finite projective dimension. We may also assume that R is complete and that M contains a minimal generator x having order ideal $O_M(x)$ of height $\leq r$. This follows from the fact that the syzygy problem remains unchanged under faithfully flat change of base as well as the preceding lemma.

Let $0 \to F_s \to \ldots \to F_1 \to F_0 \to M \to 0$ be a minimal projective resolution of M. Then the complex $F. \otimes R/I$, where $I = O_M(x)$, has homology $\mathrm{Tor}_i^R(M, R/I)$ of finite length for $i > 0$, since M is locally free on the punctured spectrum of R. Moreover, the element $x + IM$ in M/IM is nonzero, since x is not even in $\underline{m}M$, but generates a submodule of finite length, since $x \in IM$ on the free locus of M. It follows that the zero-th homology of $F. \otimes R/I$, namely M/IM, has depth zero.

It remains to show that s is at least $\dim R/I$. For if $s \geq \dim R/I$, then $\mathrm{pd}.M = s \geq \dim R/I \geq \dim R - r$, since complete local rings are catenary. Hence we obtain the inequality $\mathrm{pd}.M + r \geq \dim R$. On the other hand one has that $\mathrm{pd}.M + k \leq \dim R$ which together with the previous inequality gives that $r \geq k$ as desired. The second inequality $\mathrm{pd}.M + k \leq \dim R$ actually can be improved to $\mathrm{pd}.M + k \leq \mathrm{depth}\ R \leq f$, where $f = \min\{\dim R/P \mid P \in \mathrm{Ass}\, R\}$. We isolate this last step as a separate lemma (called by some the "New New" Intersection Conjecture). Note that S plays the role of R/I in the lemma. □

LEMMA. *Let* S *be a local ring containing a field. Let* $\mathbb{F}. : 0 \to F_s \to \ldots \to F_0 \to 0$ *be a complex of finitely generated free* S-*modules such that* $H_i(\mathbb{F}.)$ *is of finite length for* $i > 0$ *and* $H_0(\mathbb{F}.)$

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has a minimal generator which generates a nonzero submodule of finite length. Then s ≥ dim S.

Remark. Note that by the Lemme d'Acyclicité, if s < depth S, $\mathbb{F}.$ would be exact and then in turn too short to be a free resolution of the module $H_0(\mathbb{F}.)$ of depth zero. Our proof merely has to strengthen the inequality in the Lemma d'Acyclicité from depth S to dim S.

Proof of the Lemma. Suppose in fact that s < dim S. Let C be a maximal Cohen-Macaulay module over S. Then we obtain a complex $\mathbb{F}.\otimes C$ with $H_i(\mathbb{F}.\otimes C)$ annihilated by an $\underline{m}$-primary ideal for i > 0. Moreover, if c is any element in $C - \underline{m}C$, then $x \otimes c$ represents an element in $H_0(\mathbb{F}.\otimes C) - \underline{m}H_0(\mathbb{F}.\otimes C)$ and consequently is nonzero. Moreover, $x \otimes c$ is annihilated by an $\underline{m}$-primary ideal since $x \in H_0(\mathbb{F}.)$ has this property. Thus $H_0(\mathbb{F}.\otimes C)$ has depth zero. Now one applies the Lemme d'Acyclicité (see [8] or [10] for an appropriate version) to obtain the contradiction that the complex is too short to have such properties. Thus it must be that s ≥ dim S. □

The final argument here is a bit brief. This is for two reasons. First, the essential details are in our original version [8]. More interestingly, the above lemma was already proved (but not stated) by Roberts [20] in his proof that, if the homology of $\mathbb{F}.$ has finite length, then s ≥ dim S. There Roberts carefully analyzed what was needed in a separate lemma. That lemma covers our case. This theorem (and the nearly identical one of Peskine and Szpiro) really is obtained from a better understanding of the 1969 Lemme d'Acyclicité. Thus, in some sense, the essential part of the argument was known for some time although the reduction of the question to that case was not apparent (to us) until after our original proof. Perhaps more importantly, this result gives yet another application of this circle of ideas to commutative ring theory. In particular the proof of this case is made no simpler if we start with a Cohen-Macaulay (or even regular) local ring while some of the earlier applications of this technique were already understood in such cases. Thus one is enticed to look for more applications.

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