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London Mathematical Society Lecture Note Series. 94

Representations of General Linear Groups

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The right of the University of Cambridge to print and sell all manner of books was granted by Henry VIII in 1534. The University has printed and published continuously since 1584.

CAMBRIDGE UNIVERSITY PRESS

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London New York New Rochelle
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CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521269810

First published 1984
Re-issued in this digitally printed version 2007

A catalogue record for this publication is available from the British Library

Library of Congress Catalogue Card Number: 83-25171

ISBN 978-0-521-26981-0 paperback

Contents

ABSTRACT

This essay concerns the unipotent representations of the finite general linear groups $GL_n(q)$. An irreducible unipotent representation is, by definition, a composition factor of the permutation representation of $GL_n(q)$ on a Borel subgroup, and the ordinary irreducible unipotent representations may be indexed by partitions λ of n, as may the ordinary irreducible representations of the symmetric group $\mathfrak{S}_n$. The remarkable feature is that the representation theory of $\mathfrak{S}_n$ over an arbitrary field appears to be the case "q = 1" of the subject we study here.

The most important results are undoubtedly the Submodule Theorem (Chapter 11) and the Kernel Intersection Theorem (Chapter 15), but there seems to have been no previous work on the representation modules for the unipotent representations of $GL_n(q)$, so we claim originality for all the results apart from those whose source is quoted or which are obviously known (Chapters 3 - 8).

Chapters 1 and 2 set the scene, by outlining the connection between $\mathfrak{S}_n$ and representations of $GL_n(q)$ over fields of characteristic dividing q, and by giving examples of the situation to be considered later. The preliminary results which we need are derived in Chapters 3 - 8. Thereafter, we assume that the characteristic of our ground field K does not divide q, but otherwise K is arbitrary. Certain idempotents of the group algebra are defined in Chapter 9, and they are used in Chapter 10 to describe the structure of the permutation module M_λ of $GL_n(q)$ on a parabolic subgroup.

In Chapter 11, we define a certain submodule S_λ of M_λ in terms of a generator; S_λ may be regarded as the q-analogue of a Specht module. The Submodule Theorem states that every $KGL_n(q)$-submodule of M_λ either contains S_λ or is contained in $S_\lambda^\perp$. We proved the Submodule Theorem for $\mathfrak{S}_n$ in 1976 (James $[J_1]$), and thereby gave the first construction of the irreducible representations of $\mathfrak{S}_n$ over an arbitrary field. We have already published a

proof of the Submodule Theorem for $GL_n(q)$ (James [J_9]), but the proof given here is new; it is simplified by assuming initially that the ground field contains all the p^{th} roots of unity (where q is a power of p). The Submodule Theorem gives us an irreducible unipotent representation of $GL_n(q)$ for each partition of n. In particular, the various modules S_λ are the ordinary irreducible unipotent representations when the set of rational numbers is the ground field.

The aim of the next few chapters is to construct a basis for S_λ, and to prove the Kernel Intersection Theorem (Chapter 15), which describes S_λ as the intersection of the kernels of certain $KGL_n(q)$-homomorphisms defined on M_λ. Here we roughly follow the approach we adopted in 1977 (James [J_4]) to prove similar results for Specht modules. Unlike the situation for symmetric groups, where bases for Specht modules and the Kernel Intersection Theorem are easy for many special cases, the only partitions for which the $GL_n(q)$ results are clear are (n) (when there is nothing to prove!) and (n - 1, 1). Even the partition (2, 2) of 4 is difficult to handle; in place of a 2-dimensional representation of $\mathfrak{S}_4$, we have to deal with a $(q^2 + q^4)$-dimensional representation of $GL_4(q)$.

Many important results (Chapter 16) follow from the Kernel Intersection Theorem. For example, dim S_λ is shown to be independent of K, and we prove that we have found all the irreducible unipotent representations over K. The Branching Theorem, describing the structure of S_λ as a $KGL_{n-1}(q)$-module, is also deduced.

By combining the Submodule Theorem and the Kernel Intersection Theorem, it is possible to embark upon the task of finding the decomposition matrices of $GL_n(q)$ for primes which do not divide q. The problem of determining the decomposition matrices of $\mathfrak{S}_n$ is still open, and we believe that the key may well lie with the unipotent representations of $GL_n(q)$.

In Chapter 17, we prove a theorem on the decomposition matrix of

$GL_n(q)$ concerning the removal of the first column from the diagram [λ]; the corresponding $\mathfrak{S}_n$ result was proved only recently (James [J_8]).

As far as we know, only the parts of the decomposition matrix of $\mathfrak{S}_n$ corresponding to hook partitions or to two-part partitions is known (Peel [P] and James [J_2, J_3]), although work is in progress on the partitions (n - m - 1, m, 1). An analogue of Peel's results is given in Chapter 16, and in the final two chapters we determine the part of the decomposition matrix of $GL_n(q)$ which corresponds to two-part partitions, for all primes which do not divide q; the evidence that the modular representation theory of $\mathfrak{S}_n$ is just the case "q = 1" is then overwhelming.

Naturally, we have pondered the question why the modular representations of $\mathfrak{S}_n$ look like representations of the group of automorphisms of an n-dimensional vector space over "the field of one element". It is easy to be misled into giving an unsound argument about this, and it must be noted that our proofs do not translate into proofs for $\mathfrak{S}_n$. More challenging still is the explanation of the possible result that the representation theory over $\mathbb{F}_r$ of $GL_d(r)$ ($d \geq n$, r prime) is the case "q = 1" of our work here - see the remarks at the end of Chapter 16. Why should the representation theory of $GL_n(q)$ over fields whose characteristic does not divide q throw light on the representation theory of general linear groups of different dimension over fields of the natural characteristic?

Knowledge of the theory for $\mathfrak{S}_n$ has guided us to search for proofs to present here which would translate immediately into proofs for the symmetric group. We have been unsuccessful, so we cannot explain why "putting q = 1" works, and entirely new techniques have had to be developed in this essay.

LIST OF SYMBOLS

Symbol	Meaning	Chapter of definition
A_r	A certain subgroup of U^-	8.1
$A_r^{(i)}$	A certain idempotent of $\bar{K}A_r$	9.8
$B^{\pm}$	The group of upper/lower triangular matrices	5
c	A function from Γ to $\{1, 2, \ldots, q\}$	9.1
c_r	A certain function from $\Gamma(r)$ to $\{1, 2, \ldots, q\}$	9.4
E	An idempotent in $\bar{K}G_n$	
E_r	An idempotent in $\bar{K}G(\Gamma(r))$	9.4
E_λ	An idempotent in $\bar{K}U^-$	11.4
e	The exponent of q modulo p	19.4
$e_1, e_2, \ldots$	A basis for V	2.4
$\mathbb{F}_q$	The field of q elements	2.4
$(\mathbb{F}_q, +)$	The additive group of $\mathbb{F}_q$	
G_r	A subgroup of $GL_n(q)$, isomorphic to $GL_r(q)$	8.1
G_r^*	A certain subgroup containing G_r	8.1
$G(\Gamma)$	$\langle X_{ij} \mid (i, j) \in \Gamma \rangle$	5.1
$GL_n(q)$	The group of automorphisms of V	2.4
H	The group of diagonal matrices	5
H_r^*	A certain subgroup contained in G_r^*	8.1
h	The number of non-zero parts of λ	4
$h_i(\beta)$	A certain diagonal matrix	5
h_{ij}	The hook length of the (i, j) node in $[\lambda]$	20.1
I_r	The identity $r \times r$ matrix	
K	A field of characteristic coprime to q	
$\bar{K}$	K extended by a primitive p^{th} root of unity, where q is a power of p	8
$\ell_p(m)$	The least non-negative integer i such that $m < p^i$	19.1

M_λ	The permutation module on P_λ	10.1
$[m]$	$1 + q + q^2 + \ldots + q^{m-1}$	2.5
$\{m\}$	$[1]\ [2] \ldots [m]$	10.17
n	The dimension of V	2.4
$\begin{bmatrix} n \\ m \end{bmatrix}$	A Gaussian polynomial	2.14
$\begin{Bmatrix} n \\ m \end{Bmatrix}$	A binomial coefficient	
P_λ	A parabolic subgroup	6
p	A prime number	
$\mathbb{Q}$	The field of rational numbers	
q	A power of a prime number	
R	A subset of $\{1, 2, \ldots, h\}$	10.8
$\mathcal{R}_r$	The set of subsets of $\{1, 2, \ldots, h\}$ of cardinality r	10.8
$\mathcal{R}^*_r$	A certain subset of $\mathcal{R}_r$	10.20
S_λ	A certain submodule of M_λ	11.11
$\mathfrak{S}_n$	The symmetric group on n symbols	
T_λ	The initial λ-tableau	4.2
$U^\pm$	The group of upper/lower unitriangular matrices	5.6
$U^\pm_\lambda$	A certain subgroup of $U^\pm$	6.1
V	The n-dimensional vector space over $\mathbb{F}_q$ of which $GL_n(q)$ is the group of automorphisms	2.4
W	The group of permutation matrices	5
X_{ij}	A root subgroup	5
$x_{ij}(\alpha)$	An element of X_{ij}	5
$\mathbb{Z}$	The ring of integers	
$\alpha, \beta, \gamma, \delta$	Elements of $\mathbb{F}_q$	
Γ	A closed subset of Φ	5.1
Γ'	The "commutator" subset of Γ	5
$\Gamma(r)$	$\{(i, j) \mid n \geq i > j \leq r \leq n\}$	9.4
θ	A $\bar{K}G_n$-homomorphism	

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Cambridge University Press
978-0-521-26981-0 - Representations of General Linear Groups
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More information

κ	An element of K	
λ, μ, ν	Compositions of n	4
$\nu_p(m)$	The largest integer i such that p^i divides m	19.1
π, σ, τ	Permutations	
π_λ	A certain permutation, depending on λ	11.3
π_R	A certain permutation, depending on R	12.1
Φ	$\{(i, j) \mid 1 \leqq i \neq j \leqq n\}$	5
Φ^+	$\{(i, j) \mid 1 \leqq i < j \leqq n\}$	5.6
Φ^-	$\{(i, j) \mid 1 \leqq j < i \leqq n\}$	5.6
$\phi_1, \phi_2, \ldots$	Linear $\bar{K}$-characters of A_r	9.6
$\chi_1, \chi_2, \ldots$	Linear $\bar{K}$-characters of $(\mathbb{F}_q, +)$	9.1
χ_c	A linear $\bar{K}$-character of $G(\Gamma)$	9.1
χ_λ	The ordinary character of S_λ	1
$\psi_{d,i}$	A certain $\bar{K}G_n$-homomorphism defined on M_λ	15.1
$\trianglerighteq$	A transitive relation on the set of compositions of n	4.1
$\triangleright$	$\trianglerighteq$ but not =	4.1
$\langle \ldots \rangle$	The group, or vector space, generated by ...	
$\langle\ ,\ \rangle_\lambda$	A bilinear form on M_λ	11.1