1 INTRODUCTION

Among all classes of groups, it is arguable that the symmetric groups $\mathfrak{S}_n$ have the richest representation theory. Not only are the symmetric groups interesting in their own right - the theory of their representations is extremely elegant, and still contains many mysteries - but they can also be used in several ways to shed light on the representations of other groups, and their theory can be applied in fields as diverse as quantum mechanics and polynomial identity algebras. We hope to convince the reader that the representation theory of symmetric groups is just a special case of a far deeper, but equally interesting, topic, namely the theory of unipotent representations of the finite general linear groups $GL_n(q)$.

It is well-known that there is a close connection between representations of $\mathfrak{S}_n$ over a field F and the representations of $GL_n(F)$ over the same field F. But this is not what we shall explore here; instead, we open up a new avenue by considering representations of $GL_n(q)$ over a field K whose characteristic does not divide q.

The ordinary irreducible characters of $GL_n(q)$ have been determined by Green [G_1], but earlier work of Steinberg [S] produced one ordinary irreducible character χ_λ for each partition λ of n. It appears that no previous work has been done on constructing the representation modules for $GL_n(q)$, and we shall deal here entirely with the modules corresponding to the characters obtained by Steinberg, the so-called unipotent representations of $GL_n(q)$. We shall be working, then, with representations of $GL_n(q)$ which are indexed by partitions λ of n. The ordinary irreducible representations of $\mathfrak{S}_n$ are also indexed this way. Two striking results about the characters χ_λ of $GL_n(q)$ have been proved which already indicate an analogy with $\mathfrak{S}_n$:

1.1 THEOREM (Olsson [O]). If $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a partition of n then for the general linear group

$$\deg \chi_\lambda = q^{\sum_k (k-1)\lambda_k} (q^n - 1)(q^{n-1} - 1) \ldots (q - 1) \prod_h \frac{1}{(q^h - 1)},$$

where the product is over the hook lengths $h = (\lambda_i + \lambda'_j + 1 - i - j)$ in the the diagram $[\lambda]$.

(Olsson gives a similar formula for all the character degrees of $GL_n(q)$.)

1.2 THEOREM (Fong and Srinivasan [FS]). Assume that p is an odd prime not dividing q, and e is the least positive integer such that p divides $q^e - 1$. Then χ_λ and χ_μ are in the same p-block of $GL_n(q)$ if and only if $[\lambda]$ and $[\mu]$ have the same e-core.

Compare these results with two theorems from the representation theory of $\mathfrak{S}_n$ (We shall also denote by χ_λ the ordinary irreducible character of $\mathfrak{S}_n$ corresponding to the partition λ of n.)

1.3 THEOREM (Frame, Robinson and Thrall [FRT]). For the symmetric group

$$\deg \chi_\lambda = n(n-1) \ldots 1 \prod_h \left(\frac{1}{h}\right).$$

1.4 THEOREM (Brauer [B] and Robinson [R]). χ_λ and χ_μ belong to the same p-block of $\mathfrak{S}_n$ if and only if $[\lambda]$ and $[\mu]$ have the same p-core

What is more, there are similar results in the theory of Weyl modules. We shall not use Weyl modules here, so we shall not give a formal definition of them, but refer the reader to the relevant literature (for example, Green $[G_2]$ or James and Kerber [JK]). Suffice it to say that if F is a sufficiently large field, and V is the d-dimensional vector space over F on which $GL_d(F)$ acts in the natural way, then for each non-negative integer n and for every partition λ of n having at most d non-zero parts, there is a $GL_d(F)$-submodule W_λ, called a Weyl module, of $V^{\otimes n}$. We emphasize that W_λ is a representation module for $GL_d(F)$ over the "natural" field F.

1.5 THEOREM <u>The dimension of W_λ is independent of F, and</u>

$$\dim W_\lambda = \prod_{(i,j)\in[\lambda]} (d + j - i) \prod_h \left(\frac{1}{h}\right) .$$

Two Weyl modules W_λ and W_μ are said to be connected if there exists a sequence $\lambda = \lambda_1, \lambda_2, \ldots , \lambda_k = \mu$ of partitions such that for each i, W_{λ_i} and $W_{\lambda_{i+1}}$ have a common composition factor.

1.6 THEOREM <u>If F has characteristic p and λ, μ are partitions of the same integer, then W_λ and W_μ are connected if and only if $[\lambda]$ and $[\mu]$ have the same p-core</u>.

In fact, the theory of Weyl modules is much closer to the symmetric group than one might expect. For each λ, W_λ has a unique maximal submodule; we denote the quotient by F_λ, whereupon every irreducible polynomial representation of $GL_d(F)$ is isomorphic to some F_λ. For each partition λ of n, there is a Specht module, defined over F, for $\mathfrak{S}_n$. Most Specht modules have a unique maximal submodule; the various quotients give all the irreducible $F\mathfrak{S}_n$-modules. To illustrate the connection between $F\mathfrak{S}_n$-modules and $FGL_d(F)$-modules, and also for future reference, we now give an example.

1.7 EXAMPLE. The following matrices describe the composition factors of some Weyl modules. The entry in row λ and column μ is the number of composition factors of W_λ which are isomorphic to F_μ.

	(1,1)	(2)
(1,1)	<u>1</u>	0
(2)	<u>1</u>	1

if char F = 2

	(1,1,1)	(2,1)	(3)
(1,1,1)	$\underline{1}$	$\underline{0}$	0
(2,1)	$\underline{0}$	$\underline{1}$	0
(3)	$\underline{1}$	$\underline{0}$	1

if char F = 2

	(1,1,1)	(2,1)	(3)
(1,1,1)	$\underline{1}$	$\underline{0}$	0
(2,1)	$\underline{1}$	$\underline{1}$	0
(3)	$\underline{0}$	$\underline{1}$	1

if char F = 3

	(1^4)	$(2,1^2)$	(2,2)	(3,1)	(4)
(1^4)	$\underline{1}$	$\underline{0}$	0	0	0
$(2,1^2)$	$\underline{1}$	$\underline{1}$	0	0	0
(2,2)	$\underline{0}$	$\underline{1}$	1	0	0
(3,1)	$\underline{1}$	$\underline{1}$	1	1	0
(4)	$\underline{1}$	$\underline{0}$	1	1	1

if char F = 2

	(1^4)	$(2,1^2)$	(2,2)	(3,1)	(4)
(1^4)	$\underline{1}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	0
$(2,1^2)$	$\underline{0}$	$\underline{1}$	$\underline{0}$	$\underline{0}$	0
(2,2)	$\underline{1}$	$\underline{0}$	$\underline{1}$	$\underline{0}$	0
(3,1)	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{1}$	0
(4)	$\underline{0}$	$\underline{0}$	$\underline{1}$	$\underline{0}$	1

if char F = 3

Remember that all these matrices give information about every general linear group $GL_d(F)$ over F. The partitions are not partitions of the integer d. On the other hand, the matrix involving partitions of n contains the decomposition matrix of $\mathfrak{S}_n$ over F (James [J_7]); the underlined entries give this decomposition matrix. It can also be proved (James [J_{10}]) that every column of the matrix for n corresponds in a natural way to an indecomposable module for $F\mathfrak{S}_n$.

The results which are illustrated in the example above are true in general. If we had complete information about the components of permutation modules of symmetric groups, then we could construct all the matrices which describe the composition factors of Weyl modules, and conversely, since these matrices contain the decomposition matrices of symmetric groups, knowledge of the composition factors of Weyl modules would give the decomposition matrices of symmetric groups.

We hope now to have given sufficient support to the claim that the representation theory of $\mathfrak{S}_n$ is very closely tied to that of $GL_n(q)$ over fields whose characteristic divides q. How then does the representation theory of $GL_n(q)$ over fields of characteristic not dividing q fit into the picture? We aim to show that the representation theory of $\mathfrak{S}_n$ must be "the case q = 1" of this theory. It is well-known that $\mathfrak{S}_n$ looks like "the general linear group over the field of one element", but the results go through in an unexpected and beautiful way. We emphasise that the methods used in this essay do not apply to the symmetric groups; everything we achieve is analogous to the theory of symmetric groups, but whenever we suggest that q should be put equal to 1, we do so for the results but not for the proofs. We do not know why this works.

If K is a field whose characteristic does not divide q, we shall produce an irreducible $KGL_n(q)$-module for each partition of n. In general, there are fewer irreducible $K\mathfrak{S}_n$-modules than there are partitions of n, but there is a Weyl module for each partition of n. Could our new theory subsume the theory of Weyl modules, too? The evidence is flimsy, but consistent (see Chapter 16).

A light-hearted observation concerns the tendency of theorems from the representation theory of symmetric groups to take a form which does not rely on the primeness of the field characteristic (for example, several theorems involve the p-core of a diagram, which exists whether or not p is prime.) It appears that this phenomenon occurs because, for example, there are primes which divide $1 + q + q^2 + q^3$ but not $1 + q$; when we put $q = 1$ the general linear group results for these primes turn into empty theorems about those primes dividing 4 and not 2.

It will be clear by the end of this essay that there are still many interesting open questions. A recent volume on the representation theory of $\mathfrak{S}_n$ by the present author and Adalbert Kerber [JK] contained nigh on a thousand references to works on that subject. We believe that this is just the tip (q = 1) of a very big iceberg!

2 EXAMPLES

Few of the results from this section will be used later on, but it is enlightening to look at some special cases of the problems we shall consider, before plunging into the general situation.

First we consider a representation of $GL_n(q)$, denoted by $(n - 2,2)$, which looks relatively straightforward. It is already difficult, though, to complete the relevant calculations for this representation.

The symmetric group $\mathfrak{S}_n$ is the group of all permutations of $\{1, 2, ..., n\}$. We define a representation of $\mathfrak{S}_n$ which is indexed by $(n - 2,2)$, and then show what happens for the general linear group.

Consider a vector space, over a field K, whose basis elements are the unordered pairs $\{i, j\}$ from $\{1, 2, ..., n\}$. Thus, for example,

$$\kappa_1\{1, 2\} + \kappa_2\{3, 4\} \qquad (\kappa_1, \kappa_2 \in K)$$

belongs to our vector space. Since $\mathfrak{S}_n$ permutes the pairs $\{i, j\}$, our vector space may be regarded as a $K\mathfrak{S}_n$-module. The dimension of the space is simply the number of unordered pairs,

$$\frac{n(n - 1)}{2}.$$

Let $S = S_{(n-2,2)}$ be the subspace consisting of those vectors satisfying the following two conditions:

(0) The sum of the coefficients is zero. (Thus we require that $\kappa_1 + \kappa_2 = 0$ in our example above, for this condition to hold.)

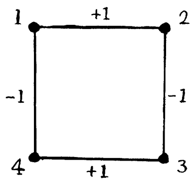
(1) For each 1-element subset $\{u\}$ of $\{1, 2, ..., n\}$, the sum of the coefficients of the pairs containing u is zero.

The subspace S is a $K\mathfrak{S}_n$-module, since conditions (0) and (1) are preserved under the action of $\mathfrak{S}_n$. An example of a vector satisfying conditions (0) and (1) is:

(2.1) $\{12\} - \{23\} + \{34\} - \{41\}$,

for certainly the sum of the coefficients $(+1 - 1 + 1 - 1)$ is zero, and if we look at $u = 2$, for example, the pairs containing u are $\{12\}$ and $\{23\}$, and the sum of the coefficients $(+1 - 1)$ of these pairs is zero. That conditions (0) and (1) are fulfilled is seen most easily by drawing a picture to describe the vector (2.1):

(2.2)



The following are facts about the space S:

(2.3) (i) S is non-zero if and only if $n \geq 4$.

(ii) S has a basis consisting of vectors of the form (2.2).

(iii) Dim $S = n(n - 3)/2$.

(iv) If K has characteristic zero, then S is an irreducible $K\mathfrak{S}_n$-module.

The first three of these results are true regardless of the field K.

We now turn to the general linear group, and choose some notation which will remain in force throughout this work:

2.4 DEFINITIONS. Let $\mathbb{F}_q$ be the field of q elements, and let V be a vector space over $\mathbb{F}_q$, with basis $e_1, e_2, \ldots, e_n$. We denote the general linear group $GL_n(q)$ by G_n. By definition, G_n is the group of all automorphisms of V. We shall freely identify G_n with the group of non-singular $n \times n$ matrices over $\mathbb{F}_q$, the automorphism given by the matrix (g_{ij}) being the one for which

$$e_i \mapsto \sum_{j=1}^{n} g_{ij}e_j \qquad (1 \leq i \leq n) .$$

If $v_1, v_2, \ldots, v_k$ are vectors in V, we let

$\langle v_1, v_2, \ldots, v_k \rangle$

denote the subspace of V spanned by $v_1, v_2, \ldots, v_k$.

2.5 DEFINITION. If m is a non-negative integer, let

$$[m] = 1 + q + q^2 + \ldots + q^{m-1} .$$

In particular, [0] = 0, [1] = 1, and $[m] = (q^m - 1)/(q - 1)$.

Note that if we put q = 1 in this definition, we get [m] = m. Bear this in mind while we go through a "q-analogue" of our statements about $\mathfrak{S}_n$.

Consider a vector space over K whose basis elements are the 2-dimensional subspaces of V. Thus, for example,

$$\kappa_1\langle e_1, e_2\rangle + \kappa_2\langle e_3, e_4\rangle \qquad (\kappa_1, \kappa_2 \in K)$$

belongs to our vector space. Since G_n permutes the 2-dimensional subspaces of V, our vector space may be viewed as a KG_n-module. The dimension of this space is simply the number of 2-dimensional subspaces of V, which, by a simple calculation (see Theorem 3.1) is

$$\frac{[n][n-1]}{[2]} .$$

Consider the subspace $S = S_{(n-2,2)}$ consisting of those vectors satisfying the following two conditions:

(2.6) (0) The sum of the coefficients is zero. (Thus, we require that $\kappa_1 + \kappa_2 = 0$ in our example above, for this condition to hold.)

(1) For each 1-dimensional subspace U of V, the sum of the coefficients of the 2-dimensional subspaces containing U is zero.

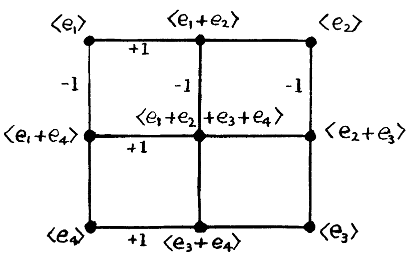
The subspace S is a KG_n-module, since conditions (0) and (1) are preserved under the action of G_n.

We recommend at this stage that the reader attempts for himself to construct a non-zero element of S. It is conceivable (and after some effort, the reader might believe likely!) that S is the zero subspace; this would not be very interesting.

To exhibit a non-zero element of S, we resort to the notation of projective geometry. A line will denote a 2-dimensional subspace U of V, and the points on the line will be the 1-dimensional subspaces of U.

EXAMPLE. If $q = 2$, $\langle e_1, e_2 \rangle$ contains three 1-dimensional subspaces $\langle e_1 \rangle$, $\langle e_2 \rangle$, and $\langle e_1 + e_2 \rangle$. In the picture below, the line through the points $\langle e_1 \rangle$, $\langle e_2 \rangle$, $\langle e_1 + e_2 \rangle$ is labelled +1; this denotes the fact that in our linear combination of 2-dimensional subspaces, $\langle e_1, e_2 \rangle$ occurs with coefficient +1.

(2.7)



The picture corresponds to

$$\langle e_1, e_2 \rangle + \langle e_1 + e_4, e_2 + e_3 \rangle + \langle e_3, e_4 \rangle - \langle e_1, e_4 \rangle - \langle e_1 + e_2, e_3 + e_4 \rangle - \langle e_2, e_3 \rangle ,$$

an element of the vector space over K whose basis elements are the 2-dimensional subspaces of V. A glance at the picture shows that conditions 2.6 hold, so we have constructed an element of S.

EXAMPLE. If $q = 3$, $\langle e_1, e_2 \rangle$ contains four 1-dimensional subspaces, $\langle e_1 \rangle$, $\langle e_2 \rangle$, $\langle e_1 + e_2 \rangle$, $\langle e_1 - e_2 \rangle$.