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CHAPTER 1

PARTITION OF A SET INTO SETS OF SMALLER DIAMETER

§1. THE DIAMETER OF A SET

Consider a disc of diameter d. Any two points M and N of this disc (fig. 1) are at distance at most d, and the disc also contains two points A and B whose distance is exactly d.

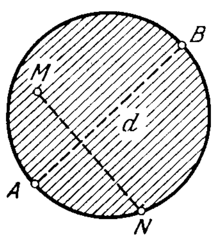

Figure 1.

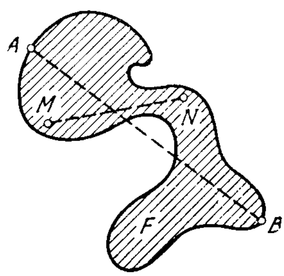

Figure 2.

Now consider another set instead of the disc. What can one call the "diameter" of this set? The observation above leads to the definition of the diameter of a set as the *greatest* distance between its points. In other words, we say that a *set* F (fig. 2) *has diameter* d *if, firstly, any two points* M *and* N *of* F *are at distance at most* d*, and secondly, one can find at least two points* A *and* B *whose distance is exactly* d (1).

For example, let F be a half-disc (fig. 3). Denote by A and B the endpoints of the semicircular arc. Then it is clear that the diameter of F is the length of the segment AB. In general, if F is a circular segment bounded by an arc ℓ and a chord a, then if the arc ℓ is not greater than a semicircle (fig. 4*a*), the diameter of F equals a (that is, the length of a chord), and if ℓ is greater than a semicircle (fig. 4*b*), then the diameter of F is the same as the diameter of the entire disc.

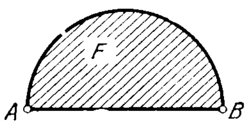

Figure 3.

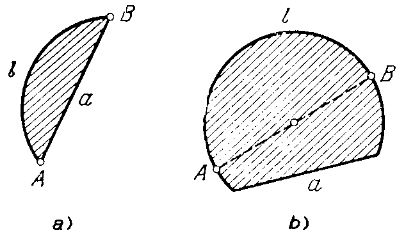

Figure 4.

It is easily seen that the diameter of a polygon F (fig. 5) is the maximal distance among its vertices. In particular, the diameter of a triangle is the length of a longest side (fig. 6).

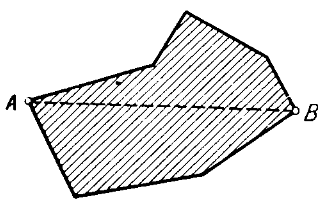

Figure 5.

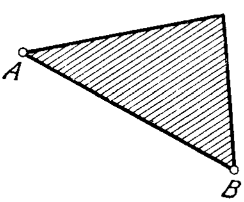

Figure 6.

Note that a set F of diameter d may contain many pairs of points at distance d. For example, an ellipse (fig. 7) contains only one such pair, a square (fig. 8) contains two pairs, an equilateral triangle (fig. 9) contains three pairs and, lastly, a disc contains infinitely many such pairs.

§2. THE PROBLEM

It is easily seen that if a disc of diameter d is partitioned into two parts by some curve MN, then at least one of these parts has diameter d. Indeed, if M' is the point diametrically opposite M, then it must belong to one of the parts, and this part (containing M

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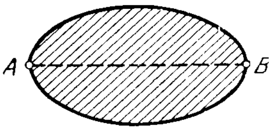


Figure 7.

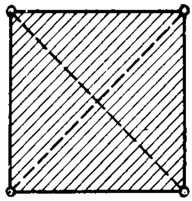


Figure 8.

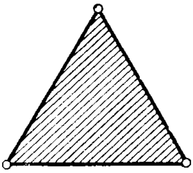
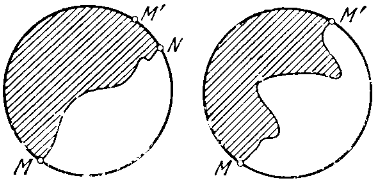
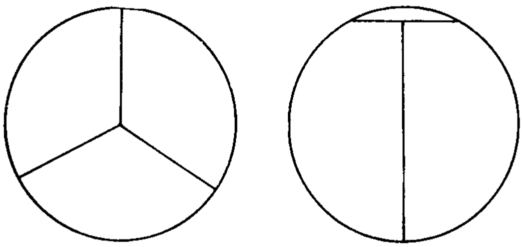


Figure 9.



a) b)
Figure 10.

and M') has diameter d (fig. 10) (2). Furthermore, it is clear that the disc can be partitioned into three parts each of diameter less than d (fig. 11).



a) b)
Figure 11.

Thus, a disc of diameter d cannot be partitioned into two parts of diameter less than d, but can be partitioned into three such parts. The same holds for an equilateral triangle of side d (for if it is partitioned into two parts, one of the parts will contain at least two vertices of the triangle, and this part will have diameter d). However, there are sets that can be partitioned into two sets of smaller diameter (fig. 12).

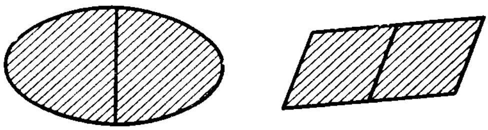


Figure 12.

Given a set F we can consider the problem of partitioning it into parts of smaller diameter (3). We denote by $a(F)$ the minimal number of sets needed in such a partition. Thus, if F is a disc or an equilateral triangle, then $a(F) = 3$, and for an ellipse or for a parallelogram we have $a(F) = 2$.

The problem of partitioning a set into sets of smaller diameter can be generalised from plane sets to bodies in three-dimensional space (or even in n-dimensional space, if the reader is familiar with this concept).

The problem of finding the possible values of $a(F)$ was posed in 1933 by the well-known Polish mathematician K. Borsuk [4]. Since then, numerous research papers have dealt with this problem. The results obtained are presented in the first chapter of this book.

Firstly we shall consider plane sets, then present a solution for three-dimensional bodies and, finally, we review the results in the n-dimensional case for the well-prepared reader.

§3. A SOLUTION OF THE PROBLEM FOR PLANE SETS

We have seen that $a(F)$ is 2 for some plane sets, and is 3 for some others. The question arises whether one can find a plane set F with $a(F) > 3$, that is, a set for which there is no partition into three parts of smaller diameter, and one has to use four or more parts. It turns out that three parts indeed always suffice, that is, we have the following theorem, proved by Borsuk in 1933 [4].

Theorem 1. *Given a plane set F of diameter d,* $a(F) \leqslant 3$; *that is, F can be partitioned into three parts of diameter less than d.*

Proof. The main part of the proof is the following lemma, proved in 1920 by the Hungarian mathematician J. Pál [33]: *every plane set of diameter d can be surrounded by a regular hexagon whose opposite sides are at distance d* (fig. 13).

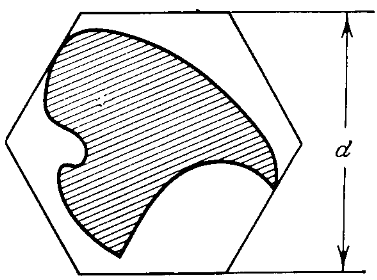


Figure 13.

Take a line ℓ that does not intersect the set F, and move it closer to F (keeping it parallel to its original direction), until it touches F (fig. 14). The resulting line ℓ_1 has at least one point in common with F, and the whole set F lies on one side of ℓ_1. Such a line is called a *support line* of F (4). Let us draw a second support line ℓ_2, parallel to ℓ_1 (fig. 14). Clearly, the whole set F will lie in the strip between the lines ℓ_1 and ℓ_2, and the distance between the lines is at most d (since the diameter of F is d).

Now draw two support lines m_1, m_2 at an angle of 60° to ℓ_1 (fig. 15). The lines ℓ_1, ℓ_2, m_1, m_2 form a parallelogram $ABCD$ with angle 60° and heights at most d, surrounding the set F.

Next draw two support lines p_1, p_2 at an angle of 120° to ℓ_1, and denote by M and N the bases of the perpendiculars dropped on these lines from the ends of the diagonal AC (fig. 15). We shall show that the direction of ℓ_1 can be chosen so that that $AM = CN$. Indeed, suppose $AM \neq CN$, say $AM < CN$. Then the value $y = AM-CN$ is negative. Now, we rotate ℓ_1 through 180° (the

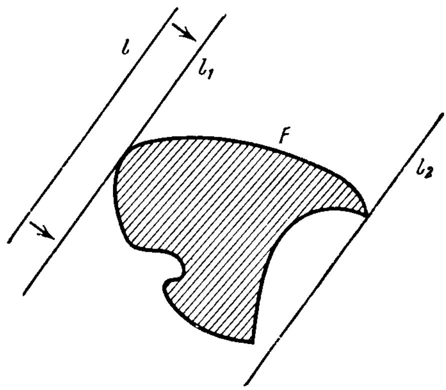


Figure 14.

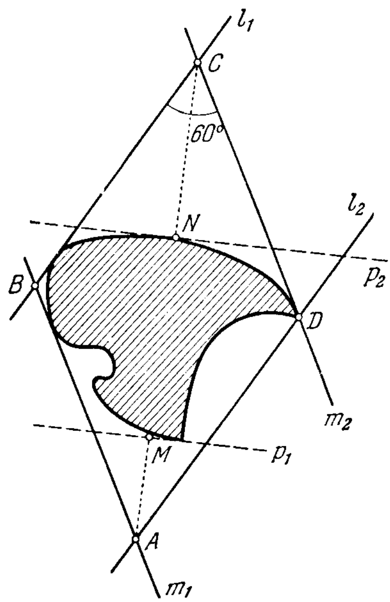


Figure 15.

set F is kept fixed). The remaining lines ℓ_2, m_1, m_2, p_1, p_2 will also change their positions (since their positions are determined by the choice of ℓ_1). Therefore, as ℓ_1 rotates, the points A, C, M, N (5) will continuously move and continuously vary the value of

$y = AM - CN$. But when the line ℓ_1 has rotated through 180°, it will lie in the position formerly occupied by ℓ_2. Hence, we shall obtain the same parallelogram as in Figure 15 with the points A and C, and also M and N, reversed. Consequently, y will be positive. If we now plot the graph of the rotation of ℓ_1 from 0° to 180° (fig. 16), we see that y is zero for some position of ℓ_1, i.e. $AM = CN$

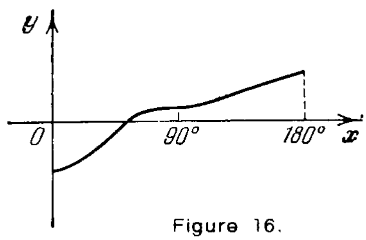


Figure 16.

(since as y continuously changes from negative to positive, it must at some point be zero). We shall examine the positions of all our lines when y is zero (fig. 17). The equality $AM = CN$ implies that

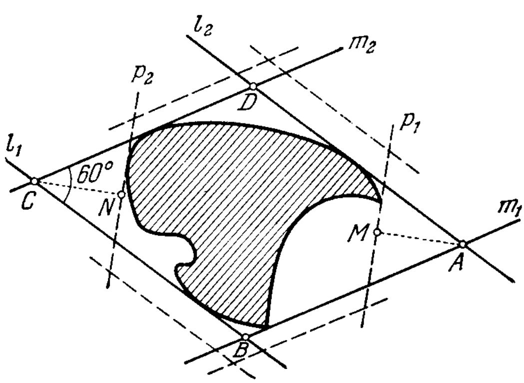


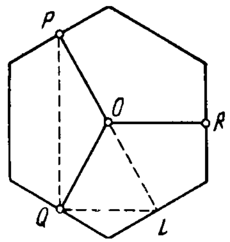
Figure 17.

the hexagon formed by the lines ℓ_1, ℓ_2, m_1, m_2, p_1, p_2 is centrally symmetric. Each angle of this hexagon is 120°, and the distance between opposite sides is at most d. If the distance between the lines p_1 and p_2 is less than d, we shall move them apart (moving each the same distance) until the distance equals d.

We then move the lines ℓ_1, ℓ_2 and m_1, m_2 in exactly the same way. We thereby obtain a centrally symmetric hexagon (with angles 120°), with opposite sides at distance d from each other (the dotted hexagon in fig. 17). From the above, it is clear that all the sides of this hexagon are equal, that is, the hexagon is regular with the set F lying inside.

Now we show that it is possible to partition this regular hexagon into three parts, each having diameter less than d. In addition, the set F will also be partitioned into three parts, each of diameter less than d. The required partition of the regular hexagon into three parts is shown in Figure 18 (the points P, Q and R are the centres of the sides, and O is the centre of the hexagon). The diameters of the parts are less than d since in the triangle PQL, the angle Q is a right-angle, and so $PQ < PL = d$.

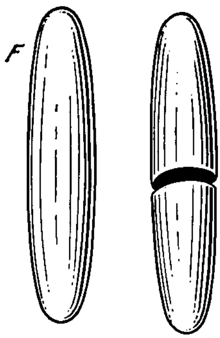
Figure 18.



This proves Theorem 1. (See Problem 4 in connection with this.)

§4. PARTITION OF A BALL INTO PARTS OF SMALLER DIAMETER

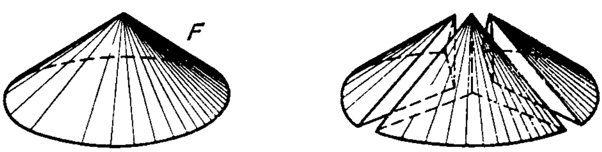
It is easily seen that in three-dimensional space there exist bodies F for which $a(F)$ equals 2 or 3. For example, if the body is very elongated in one direction (fig. 19*a*), then $a(F) = 2$ (fig. 19*b*). Furthermore, if F is a cone with height less than the radius of the base (fig. 20*a*), then $a(F) = 3$. In fact, the diameter of this body equals the diameter of the base, and therefore, $a(F) \geqslant 3$



a) b)

Figure 19.

(because it is impossible to partition even the disc at the base into two parts of smaller diameter); the partition of F into three parts of smaller diameter is shown in Figure 20*b*.



a) b)

Figure 20.

It turns out that in space, there exist bodies for which $a(F) > 3$. For example, a regular tetrahedron with side d has this property (if it is partitioned into three parts, one of the parts must contain two vertices of the tetrahedron, and therefore, the diameter of this part is d). Theorem 2 which follows shows the significantly deeper fact that a ball is also such a body.

Theorem 2. *A ball of diameter d cannot be partitioned into three parts, each of which has diameter less than d.*

Before moving to the proof, let us compare this theorem with what has already been said. (The reader not familiar with the concept of "n-dimensional" may proceed to the proof of Theorem 2 or even skip the proof and proceed directly to section §5 or Chapter 2.) As we have seen, it is impossible to partition the disc into two parts of smaller diameter. Let us call the disc a *two-dimensional ball* (two-dimensional because it lies in the plane which, as is well-known, has two dimensions). We then get the following assertion: *it is impossible to partition a two-dimensional ball into two parts of smaller diameter.* The usual ball (that is, lying in three-space) is naturally called a *three-dimensional ball*. Combining the cases of the disc and the ball, we get the following:

Theorem 2′. *For $n = 2$ or 3, it is impossible to partition an n-dimensional ball into n parts of smaller diameter.*

Apart from two-space (that is, the plane) and three-space, in mathematics and its applications, spaces of four and more dimensions are also considered. It turns out that Theorem 2′ holds not only for $n = 2$ or 3, but for an arbitrary natural number n. This theorem in its general form was proved by K. Borsuk [3] in 1932, but the essence of this result, though stated differently, was obtained even earlier (in 1930) by the Soviet mathematicians L.A. Lyusternik and L.G. Shnirel'man [32]. The proofs found by these mathematicians are highly complicated and sophisticated (they are based on theorems related to a branch of geometry called *topology*), and hence cannot be presented here. However, for $n = 3$, there is an elementary proof. (See also the theorems mentioned on page 83, proved by the German mathematician H. Lenz.)

Proof of Theorem 2. Let E be a ball of diameter d. Suppose, contrary to the assertion, that it is possible to partition E into three parts M_1, M_2, M_3, each of which has diameter less than d. Let S be the surface of the ball E. Denote by N_1 the set of all points of S belonging to M_1, and define N_2 and N_3 analogously. The sphere S is thus partitioned into three parts N_1, N_2, N_3, each of which clearly has diameter less than d. Let d_1 be the diameter