GROUP THEORY AND PHYSICS

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BASIC DEFINITIONS AND EXAMPLES

1.1 Groups: definition and examples

In this chapter we will introduce the basic mathematical concepts associated with symmetry: the notion of a group and the action of a group on a set. A group G is a set on which we are given a binary operation which behaves much like ordinary multiplication; that is, we are given a map of $G \times G \to G$ sending the pair (p, q) into pq, satisfying the associative law, the existence of an identity element e, and the existence of an inverse. That is, we assume that

- $(pq)r = p(qr)$ for any three elements p, q, r in G;
- there exists an element, e, in G such that $ep = pe = p$ for all p in G; and
- for every p in G there is a p^{-1} in G such that $pp^{-1} = p^{-1}p = e$.

Example 1

(a) Let $\mathbb{Z}_4$ denote the additive group of the integers modulo 4. The elements of this group are equivalence classes which we shall call e, a, b and c:

$$\begin{aligned} e &= \{0, \ 4, \ -4, \ 8, \ -8, \ \ldots\} \\ a &= \{1, \ 5, \ -3, \ 9, \ -7, \ \ldots\} \\ b &= \{2, \ 6, \ -2, \ 10, \ -6, \ \ldots\} \\ c &= \{3, \ 7, \ -1, \ 11, \ -5, \ \ldots\}. \end{aligned}$$

The binary operation is addition modulo 4; for example, since $1 + 3 = 4$, which equals 0 modulo 4, we have $ac = e$. The identity element is e. Since $ac = e$, $a^{-1} = c$ and $c^{-1} = a$; since $bb = e, b^{-1} = b$.

(b) Let G denote the following set of four 2×2 real matrices:

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad b = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad c = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The binary operation is matrix multiplication; for example,

$$ac = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = e.$$

The identity element is the identity matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and the inverse of each element is its matrix inverse; for example,

$$a^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = c.$$

(c) Let C_4 denote the group of rotational symmetries of the square, as follows:

$e =$ identity (rotation through 0)
$a =$ counterclockwise rotation through $\pi/2$
$b =$ counterclockwise rotation through π
$c =$ counterclockwise rotation through $3\pi/2$ (clockwise rotation through $\pi/2$).

Now the group operation is composition of transformations. Clearly the 'multiplication table' is the same as in the preceding two examples; we have considered three different realizations of the same abstract group, the so-called 'cyclic group of four elements'. It is a simple example of a *finite* group.

Example 2

We turn now to an example of a group which has an infinite number of elements. Let $SL(2, \mathbb{C})$ denote the set of 2×2 matrices of determinant 1 with complex entries. Thus, an element A of $SL(2, \mathbb{C})$ is given as

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where a, b, c and d are complex numbers satisfying

$$ad - bc = 1.$$

Multiplication is the ordinary multiplication of matrices. Since the determinant of the product of two matrices is the product of their determinants, we see that if A and B are elements of $SL(2, \mathbb{C})$, then so is their product AB. If A is an element of $SL(2, \mathbb{C})$, so that $\det A = 1$, then A is invertible and $\det A^{-1} = 1$, so that A^{-1} exists and lies in $SL(2, \mathbb{C})$. The identity element of the group is the identity matrix, i.e.

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The associative law holds for matrix multiplication and thus $SL(2, \mathbb{C})$ is indeed a group. Notice that the commutative law does not hold in general for this group.

More generally, we can consider $n \times n$ matrices with either real or complex entries. The collection of real invertible $n \times n$ matrices is denoted by $GL(n, \mathbb{R})$. (Notice that here the condition of invertibility has to be added as a supplemental hypothesis. Not all

2×2 or $n \times n$ matrices are invertible, but those that are invertible form a group.) The group $GL(n, \mathbb{R})$ is called the *real general linear* group in n variables. We can also consider the group $SL(n, \mathbb{R})$ consisting of the $n \times n$ real matrices of determinant 1. It is called the *real special linear* group in n variables. Similarly, we can consider the group $GL(n, \mathbb{C})$ of all invertible complex $n \times n$ matrices or the group $SL(n, \mathbb{C})$ of all $n \times n$ complex matrices of determinant 1.

Example 3

As a third example of a group we can consider the group, $O(3)$, of all orthogonal transformations in Euclidean three-dimensional space. This is the group of all linear transformations of three-dimensional space which preserve the Euclidean distance; that is, those transformations, A, which satisfy

$$\| A\mathbf{v} \| = \| \mathbf{v} \|$$

for all vectors $\mathbf{v}$ in ordinary three-dimensional space. If we choose an orthonormal basis for three-dimensional space so that every A becomes identified with a matrix, then A is an orthogonal transformation if and only if

$$AA^{\mathrm{t}} = e,$$

where e denotes the identity matrix in three dimensions. Notice that this equation is the same as $A^{\mathrm{t}} = A^{-1}$. We see immediately that the product of any two orthogonal transformations is again orthogonal and that the inverse of any orthogonal transformation exists and is orthogonal. Thus, the collection of all orthogonal transformations does indeed form a group. Since $\det A = \det A^{\mathrm{t}}$, it follows from $AA^{\mathrm{t}} = \mathrm{e}$ that $(\det A)^2 = 1$. Thus, for any orthogonal transformation A we have $\det A = \pm 1$. The collection of those matrices which are orthogonal, and which satisfy the further condition that $\det A = +1$, forms a subcollection of $O(3)$, which in itself is a group and which we will denote by $SO(3)$. We say that $SO(3)$ is a *subgroup* of $O(3)$. $SO(3)$ is called the special orthogonal group in three variables. (Similarly, $SL(n, \mathbb{C})$ is a subgroup of $GL(n, \mathbb{C})$, and $SL(n, \mathbb{R})$ is a subgroup of $GL(n, \mathbb{R})$.) More generally, if we put the standard Euclidean scalar product on the n-dimensional space $\mathbb{R}^n$, we can consider the orthogonal group $O(n)$ of all orthogonal $n \times n$ matrices and the corresponding subgroup $SO(n)$ of those orthogonal matrices with determinant 1.

Example 4

Let $\mathbb{C}^n$ denote the n-dimensional complex vector space of all complex n-tuples with its standard Hermitian scalar product, so that

$$(\mathbf{z}, \mathbf{w}) = z_1 \bar{\mathbf{w}}_1 + \cdots + z_n \bar{\mathbf{w}}_n,$$

where

$$\mathbf{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \quad \text{and} \quad \mathbf{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}.$$

Table 1.

	1	−1
1	1	−1
−1	−1	1

Table 2.

	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

A complex matrix A is *unitary* if

$$(A\mathbf{z}, A\mathbf{w}) = (\mathbf{z}, \mathbf{w})$$

for all $\mathbf{z}$ and $\mathbf{w}$ in $\mathbb{C}^n$. If we denote $\overline{A^t}$ (the complex conjugate transpose of A) by A^*, we may say that A is unitary only if $AA^* = e$. The product of two unitary matrices is unitary, and the inverse of a unitary matrix is unitary; so the collection of unitary $n \times n$ matrices forms a group which we denote by $U(n)$. Since $\det A^* = \overline{\det A}$, we see that $|\det A| = 1$ for A in $U(n)$. The subgroup of $U(n)$ consisting of those matrices which in addition satisfy $\det A = 1$ is denoted by $SU(n)$.

Thus, for example, the group $SU(2)$ consists of all 2×2 matrices of the form

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \quad \text{where} \quad |a|^2 + |b|^2 = 1.$$

Example 5

We can generalize Examples 1(*a*), (*b*) and (*c*) by replacing the number 4 by any positive integer. For instance, we can consider the group C_2 consisting of two elements with the 'multiplication table' as in Table 1, which is isomorphic to the additive group of the integers modulo 2. Similarly, we can think of the three-element group, C_3, with elements $1, \omega, \omega^2$ where $\omega = \exp 2\pi\mathrm{i}/3$ which obey the 'multiplication table' shown in Table 2.

The group C_3 can be thought of as the additive group of the integers modulo 3, or as the group of all rotations in the plane which preserve an equilateral triangle centered at the origin. Thus, ω represents rotation through $2\pi/3 = 120°$.

We have already considered the group C_4 of all rotations preserving a square. It

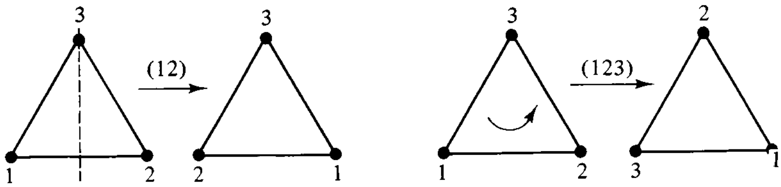

Fig. 1.1

contains four elements, consisting of the identity, rotation through $\pi/2$, rotation through π, and rotation through $3\pi/2$. We can now recognize that C_4 is a subgroup of $SO(2)$, the group of all rotations in the plane. More generally, we can consider C_n as the group of all rotations which preserve a regular polygon with n sides. It will consist of the identity and all rotations through angles of the form $2\pi k/n$.

Example 6

Let us go back to the equilateral triangle. We can consider the group of *all* symmetries of the triangle, not only the rotations. That is, we can allow reflection about perpendicular bisectors as well. This group has six elements; we will denote it by S_3. Notice that we can find some element in S_3 which has the effect of making any desired permutation of the vertices of the triangle. Let us denote the vertices of the triangle by 1, 2 and 3 (Fig. 1.1). Suppose, for example, that (12) denotes the permutation which interchanges the vertices 1 and 2 but leaves a third vertex, 3, fixed. This permutation can be achieved by a reflection about the perpendicular bisector of the edge joining 1 to 2.

Similarly, let (123) denote the permutation that sends 1 into 2, 2 into 3, and 3 into 1. This can be achieved by rotating the triangle through 120°. The permutation (132), which sends 1 into 3, 3 into 2, and 2 into 1, is achieved by rotating the triangle through 240°. From this we see that the group of symmetries of an equilateral triangle is the same as the group of all permutations on three symbols.

Suppose we consider four symbols 1, 2, 3, 4, instead of three. Let s be a one-to-one map of this four-element set onto itself. Thus, s is a permutation of this four-element set. There are four possibilities for $s(1)$: it can be any of the numbers 1, 2, 3, 4. Once we know what $s(1)$ is, then there are three remaining possibilities for $s(2)$, then two remaining possibilities for $s(3)$. Finally, $s(4)$ will be completely determined by being the last remaining number. Thus, there are $4 \cdot 3 \cdot 2 \cdot 1 = 4! = 24$ permutations on four letters. The group S_4 is the group of all these permutations. Similarly, we define the group S_n to be the group of all permutations; that is, all one-to-one transformations on a set with n elements.

Example 7

As a final example, we consider the group of all symmetries of the square, denoted by D_4. D_4 contains eight elements: four rotations, together with four reflections – the reflections about the two diagonals, and the reflections about the two perpendicular bisectors (see Fig. 1.2).

Each element of D_4 permutes the vertices 1, 2, 3, 4 of the square. Thus, we may regard

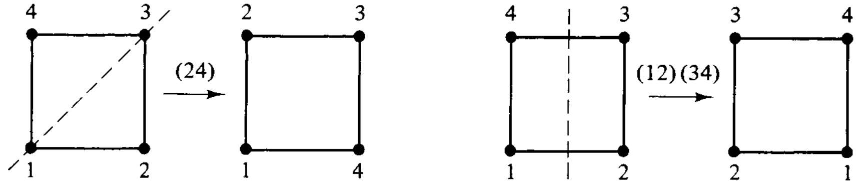

Fig. 1.2

D_4 as a subgroup of S_4, but not every element of S_4 (which has 24 elements all together) lies in D_4, which has only eight elements. Similarly, the group D_n, the group of symmetries of the regular polygon with n sides, is a subgroup of the group S_n of permutations of n symbols. The reader should check that D_n contains $2n$ elements.

1.2 Homomorphisms: the relation between $SL(2, \mathbb{C})$ and the Lorentz group

Let G_1 and G_2 be groups. Let ϕ be a map from G_1 to G_2. We say that ϕ is a *homomorphism* if

$$\phi(ab) = \phi(a)\phi(b) \quad \text{for all } a \text{ and } b \text{ in } G_1.$$

The notion of homomorphism is central to the study of groups and so we give some examples. Take $G_1 = \mathbb{Z}$ to be the integers and $G_2 = C_2$. Define the map ϕ by

$$\phi(n) = (-1)^n.$$

Recall that group 'multiplication' is ordinary addition in $\mathbb{Z}$ so that the condition that ϕ be a homomorphism reduces to the assertion that

$$\phi(a + b) = \phi(a)\phi(b),$$

i.e that

$$(-1)^{a+b} = (-1)^a(-1)^b$$

which is clearly true. More generally, we can define a homomorphism from $\mathbb{Z}$ to C_k by

$$\phi(a) = \exp 2\pi \mathrm{i} a/k = \omega^a, \quad \text{where } \omega \text{ equals } \exp 2\pi\mathrm{i}/k.$$

This generalizes the construction of Example 1 of the preceding section. Basically, what the homomorphism ϕ is telling us is that we can regard multiplication in C_k as 'addition modulo k' in the integers.

We now want to describe another homomorphism which has many important physical applications and which will recur frequently in the rest of this book. For this we need to introduce still another group, the Lorentz group. Let M denote the four-dimensional space $M = \mathbb{R}^4$, with the 'Lorentz metric'

$$\|\mathbf{x}\|^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2, \quad \text{where } \mathbf{x} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Thus, M is the ordinary Minkowski space of special relativity, where we have chosen units in which the speed of light is unity. A Lorentz transformation, B, is a linear transformation of M into itself which preserves the Lorentz metric, i.e. which satisfies

$$\|B\mathbf{x}\|^2 = \|\mathbf{x}\|^2, \quad \text{for all } \mathbf{x} \text{ in } M.$$

We let L denote the group of all Lorentz transformations; L is called the Lorentz group.

We now describe a homomorphism from the group $SL(2, \mathbb{C})$ to the group L. For this purpose we shall identify every point $\mathbf{x}$ in M with a two-by-two self-adjoint matrix, as follows:

$$x = \begin{pmatrix} x_0 + x_3 & x_1 - \mathrm{i}x_2 \\ x_1 + \mathrm{i}x_2 & x_0 - x_3 \end{pmatrix} \quad \text{represents } \mathbf{x} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Notice that

$$x^* = \overline{x^t} = x,$$

and that

$$\det x = \|\mathbf{x}\|^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2.$$

Indeed, the most general self-adjoint 2×2 matrix can be written in this form: if $x = x^*$ is a self-adjoint matrix, then its diagonal entries must be real. We can let $x_0 = \frac{1}{2}\operatorname{tr} x = \frac{1}{2}$(the sum of the diagonal entries of x) and similarly $x_3 = \frac{1}{2}$(the difference of the diagonal entries of x). Also, we can write the entry in the lower left-hand corner of x as $x_1 + \mathrm{i}x_2$. Then the entry in the upper right-hand corner will be $x_1 - \mathrm{i}x_2$. In effect, what we have done is to note that the collection of 2×2 self-adjoint matrices is a four-dimensional real vector space, for which a convenient basis consists of the identity matrix $\mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and the three so-called 'Pauli matrices'

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We have identified the vector

$$\mathbf{x} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

with the matrix

$$x = x_0 e + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3.$$

Now let A be any 2×2 matrix. We define the action of the matrix A on the self-adjoint matrix x by

$$x \to AxA^*$$

and we denote the corresponding action on the vector $\mathbf{x}$ by $\phi(A)\mathbf{x}$. Notice that $(AxA^*)^* = A^{**}x^*A^* = AxA^*$, so that AxA^* is again self-adjoint. Notice also that

$$\det(AxA^*) = |\det A|^2 \det x.$$

Therefore, if A is in $SL(2, \mathbb{C})$, then

$$\| \phi(A)\mathbf{x} \|^2 = \| \mathbf{x} \|^2,$$

so that, if A is in $SL(2, \mathbb{C})$, $\phi(A)$ represents a Lorentz transformation. Notice also that

$$ABx(AB)^* = ABxB^*A^* = A(BxB^*)A^*$$

so that

$$\phi(AB)\mathbf{x} = \phi(A)\phi(B)\mathbf{x}.$$

Thus, $\phi(AB) = \phi(A)\phi(B)$, so that ϕ is a homomorphism. Notice, however, that $\phi(-A) = \phi(A)$ so that ϕ is not one-to-one. The matrices A and $-A$ correspond to the same Lorentz transformation.

Suppose now that A belongs to the subgroup $SU(2)$ of $SL(2, \mathbb{C})$. This means that A is a unitary matrix, satisfying

$$AA^* = I; \quad \text{i.e.} \quad AIA^* = I.$$

Therefore, if $\mathbf{e}_0$ denotes the vector

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

which is represented by the 2×2 identity matrix I, then

$$\phi(A)\mathbf{e}_0 = \mathbf{e}_0.$$

If a Lorentz transformation C satisfies $C\mathbf{e}_0 = \mathbf{e}_0$, then C also carries the three-dimensional space $e_0^\perp$, consisting of vectors

$$\begin{pmatrix} 0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

into itself and C is an orthogonal transformation on that three-dimensional space. Put another way, we can regard $O(3)$ as the subgroup of L consisting precisely of those Lorentz transformations which satisfy $C\mathbf{e}_0 = \mathbf{e}_0$. Thus, the mapping ϕ, when restricted to $SU(2)$, maps $SU(2)$ into $O(3)$.

For example, let us consider the diagonal matrix

$$U_\theta = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}.$$

The $\phi(U_\theta)\mathbf{x}$ may be computed by matrix multiplication as follows:

$$U_\theta x U_{-\theta} = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

$$= \begin{pmatrix} x_0 + x_3 & e^{-2i\theta}(x_1 - ix_2) \\ e^{2i\theta}(x_1 + ix_2) & x_0 - x_3 \end{pmatrix}.$$

Thus, $\phi(U_\theta)$ leaves x_0 and x_3 unchanged and hence is a rotation about the x_3 axis. Since it sends $x_1 + ix_2$ into $e^{2i\theta}(x_1 + ix_2)$, we see that it is a rotation through angle 2θ in the x_1, x_2 plane. We have thus shown that

$\phi(U_\theta)$ is rotation through angle 2θ about the x_3 axis.

Notice that as θ ranges from 0 to π the corresponding rotation goes from 0 to 2π, making a complete circuit. As θ ranges from 0 to 2π, the corresponding rotation goes through *two* complete circuits. This is a reflection of the fact that $\phi(-A) = \phi(A)$.

Similarly, consider the action of the unitary matrix

$$V_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \quad \text{for which} \quad V_\alpha^* = V_{-\alpha}.$$

We calculate $\phi(V_\alpha)\mathbf{x}$ by matrix multiplication as follows:

$$V_\alpha x V_{-\alpha} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}.$$

We can easily determine the action of $\phi(V_\alpha)$ on the vector

$$\mathbf{e}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

by taking $x_0 = x_1 = x_3 = 0$, so that

$$x = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

We find that $\phi(V_\alpha)\mathbf{e}_2 = \mathbf{e}_2$, so that $\phi(V_\alpha)$ must be a rotation about the x_2 axis.

We now determine the action of $\phi(V_\alpha)$ on the basis vector

$$\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

by taking $x = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We find

$$V_\alpha \sigma_3 V_{-\alpha} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$$

which corresponds to the vector

$$\begin{pmatrix} 0 \\ \sin 2\alpha \\ 0 \\ \cos 2\alpha \end{pmatrix}.$$

We conclude that

$$\phi(V_\alpha)\mathbf{e}_3 = \mathbf{e}_3 \cos 2\alpha + \mathbf{e}_1 \sin 2\alpha$$

so that V_α represents rotation through angle 2α about the x_2 axis.

As a third example, consider the diagonal matrix with real entries

$$M_r = \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}.$$

Since $M_r = M_r^*$, we have

$$M_r \times M_r^* = \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \begin{pmatrix} x_0 + x_3 & x_1 + \mathrm{i}x_2 \\ x_1 - \mathrm{i}x_2 & x_0 - x_3 \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} r^2(x_0 + x_3) & x_1 + \mathrm{i}x_2 \\ x_1 - \mathrm{i}x_2 & r^{-2}(x_0 - x_3) \end{pmatrix}.$$

Thus the Lorentz transformation $\phi(M_r)$ leaves x_1 and x_2 alone whereas the x_0 and the x_3 coordinates are transformed into

$$x_0' = \tfrac{1}{2}(r^2 + r^{-2})x_0 + \tfrac{1}{2}(r^2 - r^{-2})x_3 \quad \text{and} \quad x_3' = \tfrac{1}{2}(r^2 - r^{-2})x_0 + \tfrac{1}{2}(r^2 + r^{-2})x_3.$$

We recall the definition of the hyperbolic functions:

$$\cosh u = \tfrac{1}{2}(\mathrm{e}^u + \mathrm{e}^{-u}) \quad \text{and} \quad \sinh u = \tfrac{1}{2}(\mathrm{e}^u - \mathrm{e}^{-u}).$$

The *Lorentz boost* in the z direction with parameter t, denoted by L_t, is defined as the transformation given by

$$x_1' = x_1,\ x_2' = x_2,\ x_0' = (\cosh t)\,x_0 + (\sinh t)x_3 \quad \text{and} \quad x_3' = (\sinh t)x_0 + (\cosh t)x_3.$$

In other words, L_t^z is the Lorentz transformation given by the matrix

$$L_t^z = \begin{pmatrix} \cosh t & 0 & 0 & \sinh t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh t & 0 & 0 & \cosh t \end{pmatrix}.$$

If we set $r = \mathrm{e}^t$ then our preceding computation shows that

$$\phi(M_{\mathrm{e}^t}) = L_{2t}^z.$$

To summarize: let R_θ^z denote rotation through angle θ about the z axis, let R_θ^y denote rotation through angle θ about the y axis. We have shown that

$$\phi(U_\theta) = R_{2\theta}^z,\ \phi(V_\theta) = R_{2\theta}^y \quad \text{and} \quad \phi(M_{\mathrm{e}^t}) = L_{2t}^z.$$

We might now ask what is the range of ϕ: that is, which elements C in the Lorentz group L are actually of the form $\phi(A)$ for some A in $SL(2, \mathbb{C})$? We first show that every A in $SL(2, \mathbb{C})$ can be continuously joined to the identity by a curve A_t of matrices which all lie in $SL(2, \mathbb{C})$. By a standard theorem of linear algebra we know that any A is conjugate to an upper triangular matrix, that is

$$A = B\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}B^{-1}.$$

Now simply let a_t be any curve of non-zero complex numbers with $a_0 = 1, a_1 = a$, and let b_t be any curve of complex numbers with $b_0 = 0$ and $b_1 = b$. Then

$$A = B\begin{pmatrix} a_t & b_t \\ 0 & a_t^{-1} \end{pmatrix}B^{-1}$$

is a curve of matrices in $SL(2, \mathbb{C})$, with $A_0 = I$ and $A_1 = A$. This demonstrates explicitly that any A in $SL(2, \mathbb{C})$ can be joined to the identity by a continuous curve.

However, not every element of L can be joined to the identity element by a continuous curve. For instance, there exist elements in L which are 4×4 matrices with negative determinant. As the determinant is a continuous function, there can be no curve joining an element with a negative determinant to the identity. Furthermore, every element of L preserves the set of time-like vectors, that is those vectors with $\|\mathbf{x}\|^2 > 0$. The set of time-like vectors falls into two components according to whether x_0 is positive or negative. An element of L can interchange these two components, but obviously any element of L which can be continuously joined to the identity must preserve each component. It follows that there are elements in L which cannot lie in the range of ϕ.

We shall denote by L^0 the *proper* Lorentz group, the subgroup of L consisting of those transformations which have positive determinant and which preserve the forward light cone, that is, which send each component of the set of time-like vectors into itself. It will emerge from discussions in the course of the next few sections that the mapping ϕ sends $SL(2, \mathbb{C})$ onto L^0 and sends the subgroup $SU(2)$ onto the group $SO(3)$ of rotations in three-space.

The proof goes as follows:

Lemma
Every proper Lorentz transformation, B, can be written as

$$B = R_1 L_u^z R_2,$$

where R_1 and R_2 are rotations, and L_u^z is a suitable Lorentz boost in the z direction.

Proof We can write

$$Be_0 = \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix},$$

where $\mathbf{x} = x_1e_1 + x_2e_2 + x_3e_3$ and $x_0^2 - \|\mathbf{x}\|^2 = 1$. We can find a rotation S which rotates the vector $\mathbf{x}$ to the positive z axis, so

$$SBe_0 = \begin{pmatrix} x_0 \\ 0 \\ 0 \\ \|\mathbf{x}\| \end{pmatrix}.$$

The self-adjoint matrix that corresponds to SBe_0 is thus

$$\begin{pmatrix} x_0 + \|\mathbf{x}\| & 0 \\ 0 & x_0 - \|\mathbf{x}\| \end{pmatrix}.$$

Now choose r so that $r^2 = (x_0 + \|\mathbf{x}\|)^{-1} = x_0 - \|\mathbf{x}\|$. (Remember that

$$(x_0 + \|\mathbf{x}\|)(x_0 - \|\mathbf{x}\|) = x_0^2 - \|\mathbf{x}\|^2 = 1.)$$

Then applying M_r gives $\phi(M_r)SBe_0 = e_0$. Thus $\phi(M_r)SB$ is a rotation; call it R_2. We thus have

$$\phi(M_r)SB = R_2$$

or

$$B = S^{-1}[\phi(M_r)]^{-1}R_1.$$

This is the desired decomposition with $R_1 = S^{-1}$ and $L_u^z = [\phi(M_r)]^{-1}$.

In Section 1.6 we will prove a theorem due to Euler which asserts that every rotation R in three-dimensional space can be written as a product

$$R = R_\theta^z R_\phi^y R_\psi^z$$

that is, as a rotation about the z axis, followed by a rotation about the y axis, followed by a rotation about the z axis again. (The angles θ, ϕ, ψ are called the *Euler angles* of the rotation R.) Combined with the above lemma, we conclude that every element of the proper Lorentz group can be written as a product of elements of the form L_u^z, R_θ^z and R_ϕ^y. But each of these is in the image of ϕ. So, granted Euler's theorem, we conclude that $\phi(SL(2,\mathbb{C}))$ is all of the proper Lorentz group.

1.3 The action of a group on a set

We now return to some general definitions. Let G be a group and let M be a set. We say that we have an *action* of G on M if we are given a mapping of $G \times M \to M$ sending (a, m) into am which satisfies the associative law

$$a(bm) = (ab)m$$

and

$$em = m \quad \text{for any } m \text{ in } M,$$

where e is the identity element of the group. Thus, an action of a group G on a set M is nothing other than a homomorphism from G into the group of all one-to-one

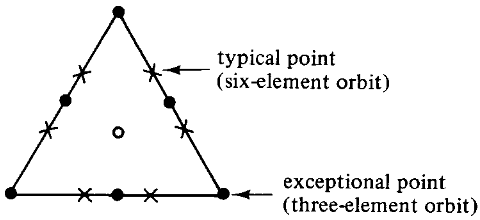

Fig. 1.3

transformations of M. Examples from the preceding section include the action of $SL(2,\mathbb{C})$ (a group) on Minkowski space (a set) and the action of the permutation group S_3 on the set of vertices of the equilateral triangle.

Let G act on M and let m be a point of M. We can consider the subset of M consisting of all points of the form am as a ranges over all elements of G. This subset of M is called the *orbit* of the point m under the action of G on M and is denoted by $G\cdot m$. Thus, $G\cdot m = \{am \mid a \in G\}$. Thus, for example, in Fig. 1.3 we have marked some orbits of various points in the triangle under the action of S_3. Notice in this example that a 'typical' point x has six points in its orbit, but there exist certain exceptional points ● and × with three-element orbits and one point ○ whose orbit consists of itself alone.

As another example, we can consider the group $SO(3)$, whose elements act as rotations in three-dimensional Euclidean space. The orbit of any point which is not the origin is the sphere centered at the origin and passing through that point, but the orbit of the origin is just one point, the origin itself.

Given any point m in M we can consider the subset of G consisting of those a in G which satisfy $am = m$. It is clear that if $am = m$, then $a^{-1}m = m$. Furthermore, if $am = m$ and $bm = m$, then $(ab)m = m$. Thus, the set of such a forms a subgroup of G, which is called the *isotropy group* of m and is denoted by G_m. Thus, for example, in the case of $SO(3)$ acting on three-dimensional space, the isotropy group of any point $m \neq 0$ will consist of those rotations which preserve the point in question, that is the subgroup of rotations about the axis passing through the point. Thus for each non-zero point m the group G_m is isomorphic to $SO(2)$ and consists of rotations in the plane perpendicular to m. Notice that any two such isotropy subgroups G_m are isomorphic as abstract groups, but are different subgroups of $SO(3)$. The isotropy group of the origin is the entire group $SO(3)$.

In the case of S_3 acting on the triangle, the isotropy group of a typical point x will consist of the identity alone. The isotropy group of a point ● with a three-element orbit will consist of the identity and one reflection. Finally, the isotropy group of the center point ○ is the entire group S_3. Notice that in all cases the product of the number of elements in the orbit by the number of elements in the isotropy subgroup is six, which is the number of elements in S_3.

This fact is true in general. Suppose that G is a finite group, and let $\#G$ denote the number of elements in G. Let m be a point of M, and consider the orbit of m under G, which contains $\#(G\cdot m)$ elements. If n is an element of this orbit, then $n = am$ for some a in G. If $n = bm$ as well, then $a^{-1}b$ must lie in G_m. This means that to each element n there are exactly $\#G_m$ group elements which map m into n. Therefore, since every element of G

carries m into some orbit element, we conclude that

$$\#G = \#(G\cdot m)\cdot \#G_m.$$

If M consists of a single orbit, we say that G acts *transitively* on M. Any time that we have a transitive group action of G on M, we can, by choosing a point m in M, determine a partition of G into classes, called *cosets*, of the form

$$aG_m = \{ab \quad \text{for all } b \text{ in } G_m\}.$$

Each coset contains the same number of elements as the subgroup G_m. The coset aG_m, which consists of all group elements that carry the point m into the point am, may be identified with the point am. Since G acts transitively on M, there is, for each element n of M, at least one element b of G such that $n = bm$. We have therefore identified each point of M with a coset of G. By introducing the notation G/G_m for the set of cosets, we may identify

$$M \quad \text{with} \quad G/G_m.$$

As an example of this identification, let M be the set of vertices of an equilateral triangle and let G be the group S_3, which permutes these vertices. We partition G into cosets by selecting vertex 1 (though 2 or 3 would have served equally well). Then the coset $\{e, (23)\}$, which is the isotropy group G_1 itself, is identified with vertex 1. The coset $\{(123), (12)\}$, consisting of elements which carry vertex 1 into vertex 2, is identified with vertex 2, and the coset $\{(132), (13)\}$ is similarly identified with vertex 3.

This identification of elements of M with cosets of G provides a useful way of constructing a set M on which G acts transitively. Suppose that H is a subgroup of G. We can form the cosets aH, each containing $\#H$ elements, so that there are $\#G/\#H$ cosets in all. We define the action of a group element g on a coset aH by $g(aH) = (ga)H$. This construction yields the same result no matter which element of the coset we choose. If, instead of a, we had chosen a different element $a' = ah$ (where $h \in H$, so that a' and a are in the same coset), then $g(a'H) = (ga')H = (gah)H = (ga)(hH) = (ga)H$ since hH is just the subgroup H. Thus we have a rule by which G acts on the space of cosets $M = G/H$, and we may label each coset in M by the name of any element of G which lies in that coset.

As an example of the construction, we construct a two-element set M on which the group S_3 acts transitively. We start with the subgroup $H = \{e, (123), (132)\}$, which corresponds to *rotational* symmetries of an equilateral triangle. The set M consists of two cosets: $\{e, (123), (132)\}$, which we call simply $[e]$, and $\{(12), (13), (23)\}$, which we call $[(12)]$. The elements of H carry each point of M into itself; the other three elements of S_3 carry $[e]$ into $[(12)]$ and $[(12)]$ into $[e]$.

1.4 Conjugation and conjugacy classes

One set M on which a group G can act is the group G itself. When G acts on itself by left multiplication, so that a transforms b into ab, then the action is always transitive. For any group elements c and b there is an element $a = cb^{-1}$ such that $ab = cb^{-1}b = c$.

Correspondingly, the isotropy subgroup of each group element, G_b, consists of the identity alone, since $ab = b$ implies that $a = e$.

There is another natural way for a group to act on itself which leads to less trivial orbits and isotropy subgroups. We say that G acts on itself by *conjugation* if a acting on b is aba^{-1}. Conjugation always defines a group action, since the action of ac on b yields

$$(ac)b(ac)^{-1} = acbc^{-1}a^{-1} = a(cbc^{-1})a^{-1},$$

which is the same as the action (by conjugation) of c followed by the action of a.

The orbits of a group under conjugation are called the *conjugacy classes* of the group. Two elements b and c belong to the same conjugacy class if there exists an element a such that

$$aba^{-1} = c.$$

From this equation follow two straightforward consequences:

(1) The identity is always a one-element conjugacy class. *Proof*: $aea^{-1} = e$ for any a.

(2) For an Abelian (commutative) group, every element is in a conjugacy class by itself. *Proof*: $aba^{-1} = aa^{-1}b = b$ in this case.

The isotropy group G_b of an element b under conjugation consists of those elements a for which $aba^{-1} = b$. One such element is $a = e$, another is b (if $b \neq e$), and a third is b^{-1} (if $b^{-1} \neq b$). The number of elements in the subgroup G_b of course obeys the general rule $\#G = \#(G \cdot b) \cdot \#G_b$ for any finite group. It follows that the number of elements in any conjugacy class is always a divisor of the number of elements of the group.

For any group of matrices, conjugate group elements B and C satisfy

$$ABA^{-1} = C.$$

It follows immediately that matrices B and C have the same eigenvalues, and hence the same determinant and trace.

As an example, we list the conjugacy classes of the group S_3. One class consists of the identity element e alone. The second class consists of the two elements (123) and (132) (note that $(23)(123)(23)^{-1} = (132)$.) Both these elements represent 120° rotations of an equilateral triangle. The third conjugacy class consists of the remaining three elements, (12), (13) and (23), which represent reflections of an equilateral triangle in the perpendicular bisector of a side.

More generally, for any permutation group, one can show that elements with the same 'cycle structure' always belong to the same conjugacy class*. For example, the group S_4 has five conjugacy classes as follows:

- the identity e (one element);
- (123), (132), (124), etc. (eight elements);
- (12), (13), etc. (six elements);
- (12)(34), (13)(24), (14)(23) (three elements);
- (1234), (1243), etc. (six elements).

* See Section 8, Chapter 2.

1.5 Applications to crystallography

The concept of an orbit of a group action is useful in clarifying the notion of 'form' in crystallography. Ordinary table salt, NaCl, if allowed to crystallize under carefully controlled conditions, forms cubic crystals. Let C be the cube in $\mathbb{R}^3$ whose vertices are the points $(\pm 1, \pm 1, \pm 1)$. The group of all orthogonal transformations which carry C into itself is denoted in crystallographic notation by O_h. It is clear that any linear transformation sending (x, y, z) into (u, v, w), where the u, v, w are permutations of $\pm x$, $\pm y$, $\pm z$, carries the cube into itself, and that these are the only linear transformations which preserve the cube. There are eight choices of $\pm$ and six possible permutations of x, y, z. The group O_h thus contains 48 elements.

NaCl can, however, crystallize in other forms. If a small amount of urea is added, then small equilateral triangles replace the corners of the cube. These triangles are congruent, so the crystal is still invariant under O_h. As more urea is added, the triangles open up into hexagons. Finally, the cube faces disappear altogether and we have an octahedron. (See Figs. 1.4(a)–(d).)

The group preserving the cube is thus the same as the group of the octahedron (which is the reason for the notation O_h).

The crystal whose appearance is as in the figure is said to show faces of two different *forms*; the eight triangular faces constitute one form whereas the six octagonal faces (modifications of the original cube faces) constitute the other form. The relative size of the various forms is called the *habit* of the crystal.

Crystals that are grown under careful conditions will develop into cubes or octahedra or other regular shapes. But if you go and pick up a crystal in the field, it will most likely have a rather irregular shape. Nevertheless, it does exhibit a symmetry, but in a more subtle sense. The first major discovery, by Nicolas Steno (1669) and Christian Huyghens in the field of crystallography is the celebrated 'law of corresponding angles' that says that the angles between 'corresponding faces' on all crystals of the same substance are equal. Put another way, suppose that we draw the normals to each of the faces, so as to get a set of points, on the unit sphere. Then (up to a rotation of the sphere) this set of points is the same for crystals of the same substance, although some of the points may be missing in a given crystal (i.e. some faces do not make their appearance). Furthermore, if a substance does (under controlled conditions)

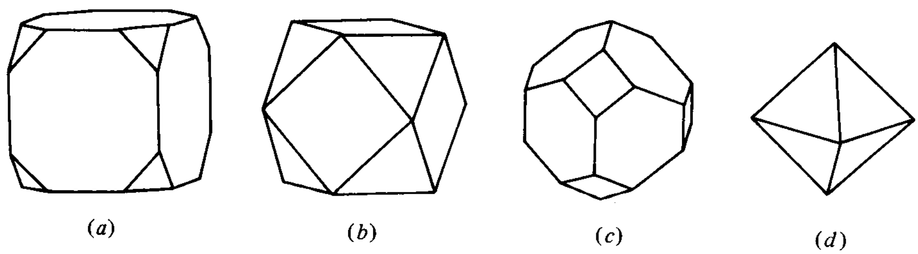

Fig. 1.4

crystallize into a regular body, then the set of normals of *any* crystal of that substance is invariant under the group of symmetries of the body. For example, suppose a substance crystallizes on occasion as a cube. Then the set of normals of any crystal of this substance will be invariant under O_h. Steno's 'law' appears in a caption to a figure in the appendix to his book entitled *De Solido intra Solidum Naturaliter Contento* (Florence, 1669).

The concept of an orbit of a group action allows us to explain in more detail what the crystallographers mean by the concept of the 'form' of a crystal. As we have already indicated, the observed symmetry of a crystal is best expressed in terms of directions. The law of the constancy of corresponding angles says that it is the collection of normal directions to the faces that is invariant (up to an overall rotation). These directions are the same for all crystals of the same substance, even though the outward appearance of the crystal might be quite irregular. We can consider these directions as points on the unit sphere. The group of symmetries of the crystal acts on this set of points. An orbit of the symmetry group acting on this set of directions is what is called a form of the crystal. To quote from a standard textbook (Phillips, *An Introduction to Crystallography*, p. 9): 'A rigid definition of a form is "the assemblage of faces necessitated by the symmetry when one face is given."'

Let us describe the possible types of orbits for the cubic group, $G = O_h$. We know that the number of elements in an orbit must divide the order of the group, which in this case is 48. It is clear that if we pick a generic point (x, y, z), with no two coordinates equal in absolute value and no coordinate zero, then the isotropy group $G_{(x,y,z)}$ consists only of the identity, and thus the orbit of this point contains the full 48 elements. The isotropy group of a point of the form (x, x, z), $|x| \neq |z|$, $x \neq 0$, $z \neq 0$, consists of the two-element permutation group – the group of transformations which permute the first two variables. The orbit through (x, x, z) thus contains 24 elements. Similarly, the orbit through $(x, x, 0)$ contains 12 elements. The orbit through (x, x, x) contains eight elements and the orbit through $(1, 0, 0)$ six elements. This last orbit corresponds to the set of faces of a cube – it is called the cubic form of the crystal. The eight-element orbit corresponds to the set of faces of the octahedron – it is called the octahedral form.

The crystallographers use the notation $\bar{x}$ for $-x$. Thus, the cubic form consists of the six elements

$$(1,0,0),\quad (\bar{1},0,0),\quad (0,1,0),\quad (0,\bar{1},0),\quad (0,0,1),\quad (0,0,\bar{1}).$$

The octahedral orbit is the orbit of the point $(1, 1, 1)$. (We write $(1, 1, 1)$ instead of $(1/3^{\frac{1}{2}}, 1/3^{\frac{1}{2}}, 1/3^{\frac{1}{2}})$, since we are really interested in the direction. We can always remember to normalize if we want the points to lie on the unit sphere.) The octahedral form thus consists of the eight points

$$(1,1,1),\quad (\bar{1},1,1),\quad (1,\bar{1},1),\quad (1,1,\bar{1}),\quad (\bar{1},\bar{1},1),\quad (1,\bar{1},\bar{1})\quad (\bar{1},1,\bar{1}),\quad (\bar{1},\bar{1},\bar{1}).$$

It is convenient to have a way of representing points of the sphere on the plane of the paper. A standard convention is to use stereographic projections. If we project from the south pole, as shown in Fig. 1.5, all points on the upper hemisphere are mapped into the