Introduction

The stellation process was described in some detail in *Polyhedron models*; so I shall simply refer you to that source for more detailed information. But I must say at once that a thorough acquaintance with this process is really a necessity if you want to acquire a deeper appreciation of its usefulness in relation to polyhedral duality. This will become very evident as you work your way through the models presented here.

Basic notions about stellation and duality

Very little has been published about stellations of Archimedean solids. My own work has led me to see that there is no reason to limit oneself to strictly Archimedean forms. In fact, some variations lead to far more aesthetically pleasing results, and such variations become necessary, unavoidably so, in the process of working out some of the dual forms of nonconvex uniform polyhedra. Interesting as the stellation process may be in its own way, it will not be the primary concern here.

As for duality, it seems strange in some ways that historically its geometrical significance was not clearly recognized until modern times. The five regular solids were known in ancient times, as was the entire set of thirteen semiregular solids. Johannes Kepler, in 1611, seems to have been the first to have recognized that the rhombic dodecahedron is the dual of the cuboctahedron. Other duals seem not to have entered into historical perspective until the work of E. Catalan, a French mathematician, who published his results in 1862. At the beginning of the twentieth century, M. Brückner summarized all the results of polyhedral research known at that time. In his classic work *Vielecke und Vielfläche* he gives an exact definition of duality, or what more strictly must be called the polar reciprocal relationship, and he exhibited many dual forms as models in his photographic plates.

What precisely is involved here? By way of a general description it may be said that the dual of any polyhedron is one that has the same number of edges as the original from which it is derived, but there is an interchange in the numbers of faces and vertices. The kinds of faces and vertices are such, however, that an n-sided polygon as face in the original yields an n-edged vertex in the dual. However, this is at best a rather vague definition or description of the duality relationship. Another way for you to picture the process by which the dual is generated is to fix your attention on the point called the incenter of a polyhedral face and then think of this point moving out from its position on the surface of the original polyhedron. The movement of this point (mathematically speaking, its translation) must take place along an axis of central symmetry of the solid. Such a translation will eventually bring the point to a position that coincides with a vertex point of the dual of the original polyhedron. But how far must this point move? The answer is given in the polar reciprocal relationship, which will be considered next.

Polar reciprocation

It will be useful first to introduce the notion of polar reciprocation in two-dimensional space, namely in plane geometry. In higher geometries, such as projective geometry, the notion of duality has some far-reaching consequences, but here only one aspect of it is needed, a very simple one indeed. Its very simplicity belies its far-reaching consequences even here. As a theorem in plane geometry, the basic idea of polar reciprocation can be found in Euclid. It is also directly related to the famous theorem of Pythagoras. Figure 1 shows a circle whose center is O, with a point P' inside it and P outside it, and Q is on the circumference of the circle. P is the polar reciprocal of P' if and only if $OP \cdot OP' = OQ^2$. If a, b, and r name the measured distances of P, P', and Q from O, then algebraically this theorem says that $ab = r^2$. The proof of the theorem is derived from the similarity of the right-angle triangles formed by joining P' to Q. Because corresponding sides of similar triangles are proportional, it follows that $a : r = r : b$, namely, $ab = r^2$.

In three-dimensional space (i.e., in solid geometry) you need only imagine Fig. 1 as representing a cross section through a sphere with P' inside and P outside the sphere, with Q lying on its surface and O being the center of the sphere, so that OQ becomes a radius of the sphere. The algebraic formula $ab = r^2$ can now be used to determine the exact distance of P as a vertex point of a dual in relation to the point P' taken here to represent the incenter of a given polyhedral face. This definition is very important, because the dual form can take on some very subtle transformations or variations if it is disregarded. For example, in the Epilogue of *Polyhedron models* I referred to the models made from Figs. 7, 8, and 9 (shown on p. 6 of that book) as Archimedean duals, which they really are not. In *Spherical models* (p. 51) I made reference to spherical duals. The three duals just mentioned really are plane or flat models of spherical duals. These are not the same as polar reciprocal duals. As another example, it might be pointed out that in

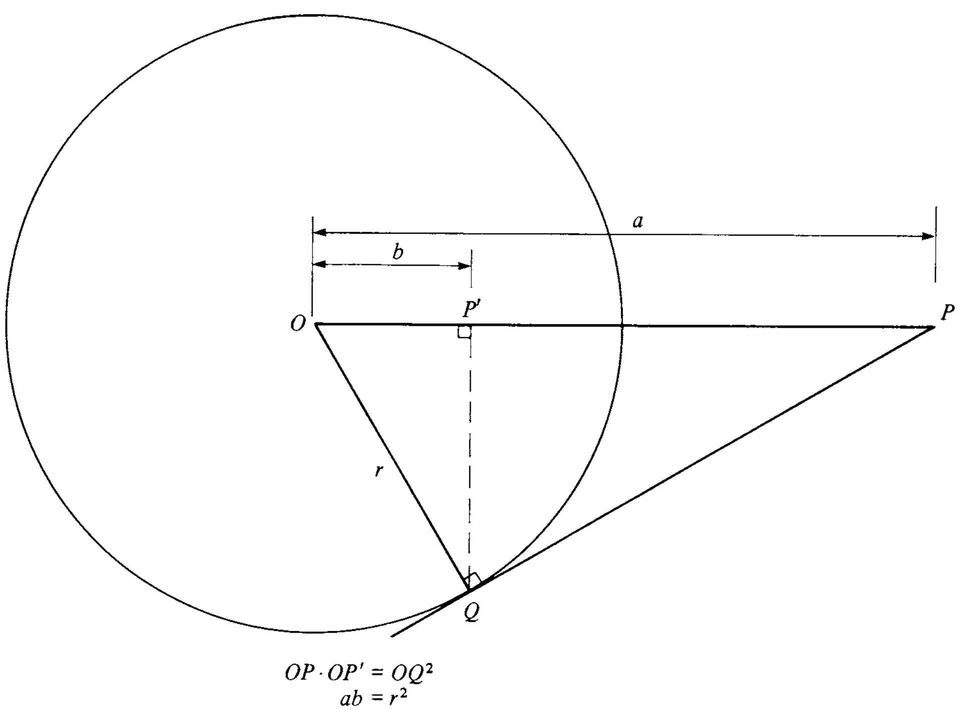


Fig. 1. Polar reciprocation in plane geometry.

a very artistically illustrated book, *Polyhedra, the realm of geometric beauty*, the author, U. Graziotti, attempted to show how the Archimedean duals can be geometrically derived by the process of erecting pyramids on the faces of a chosen basic polyhedron, the vertices of these pyramids thus determining the vertices of the related dual. Graziotti fixed his attention exclusively on the shape of the lateral faces of such pyramids and in the process neglected to consider the exact heights of such pyramids. Actual calculation involving the polar reciprocal formula disqualifies seven of the thirteen semiregular duals he intended to show.

It might be good at this time to give a fuller exposition of the polar reciprocal relationship. If you find the following presentation too abstract on a first reading, you may want to skip it for now and return to it later on, when the handling of models may greatly aid you toward a better understanding. I am indebted to H. Martyn Cundy for the following elaboration. He sent this to me after reading an article I wrote in which I enunciated the following conjecture: "The dual of any given non-convex uniform polyhedron is a stellated form of the dual of the convex hull of the given solid." The convex hull (which Coxeter calls the "case") is the smallest convex solid that can contain it. The dual of this convex hull is either known or can be found by using the polar reciprocal formula. Once it is found it serves as the "core" of the stellation process.

Here is Cundy's summary:

1. Every uniform polyhedron has all its vertices lying on a sphere.
2. The process of forming the dual is equivalent to taking the polar reciprocal in this sphere.
3. In polar reciprocation,
   every point is replaced by its polar plane
   every plane is replaced by its pole.
4. If the sphere has center O and radius r, the polar plane p of the point P is the plane normal to OP through N, where N is on OP and $OP \cdot ON = r^2$. See Fig. 2.
5. If P is on the sphere, p is a tangent plane at P. If P is outside the sphere, the tangents from P to the sphere meet it at points on p. If P is on q, the polar plane of Q, then Q is on p. See Fig. 3.
6. The plane p is between O and Q if and only if the plane q is between O and P. See Fig. 4.
7. The convex hull of a uniform polyhedron is a polyhedron whose dual is a convex polyhedron with an inscribed sphere touching all its faces.

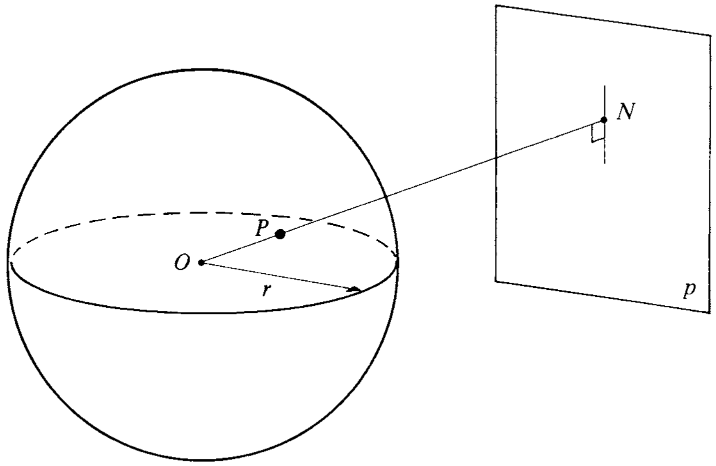


Fig. 2. Polar reciprocation in solid geometry.

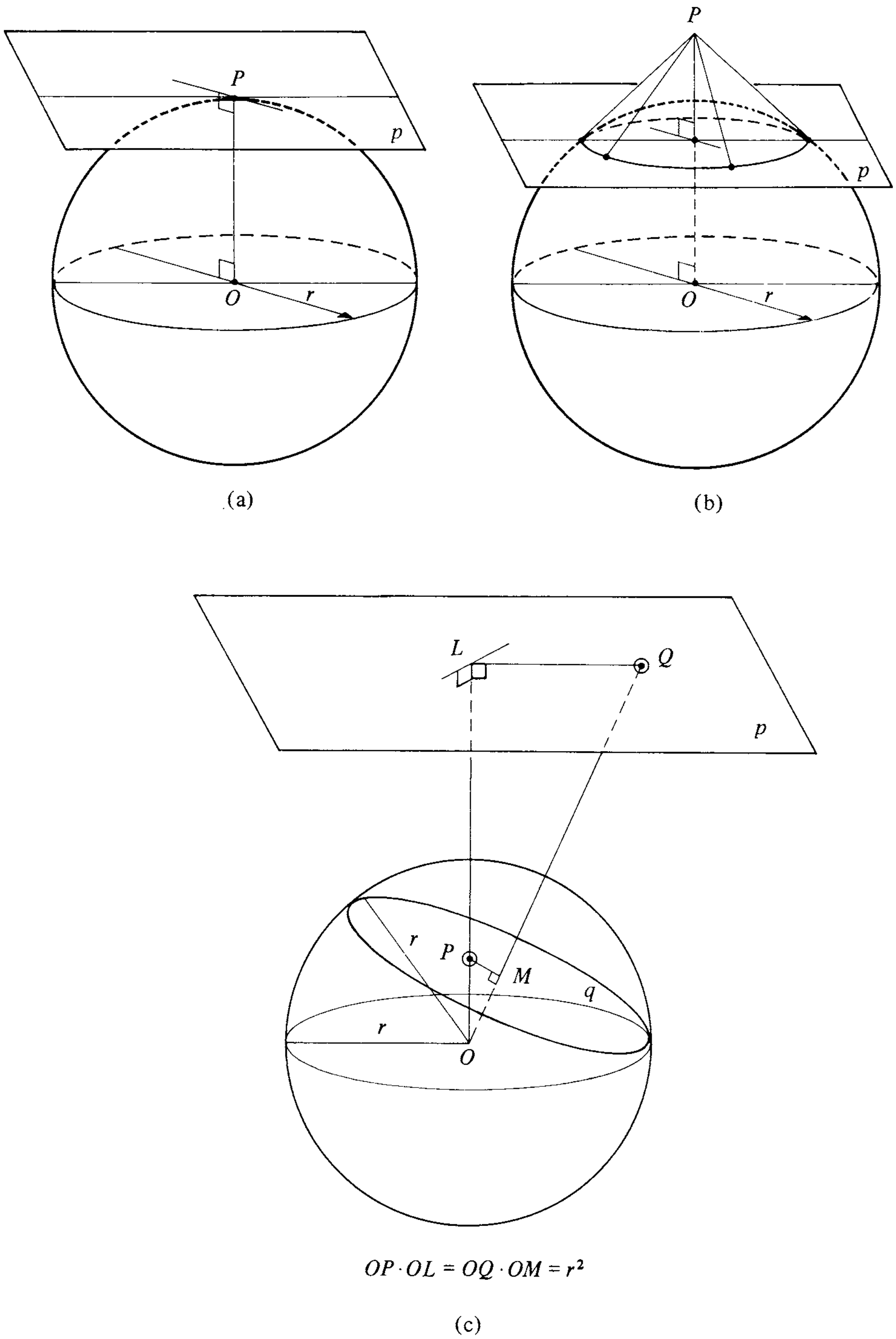
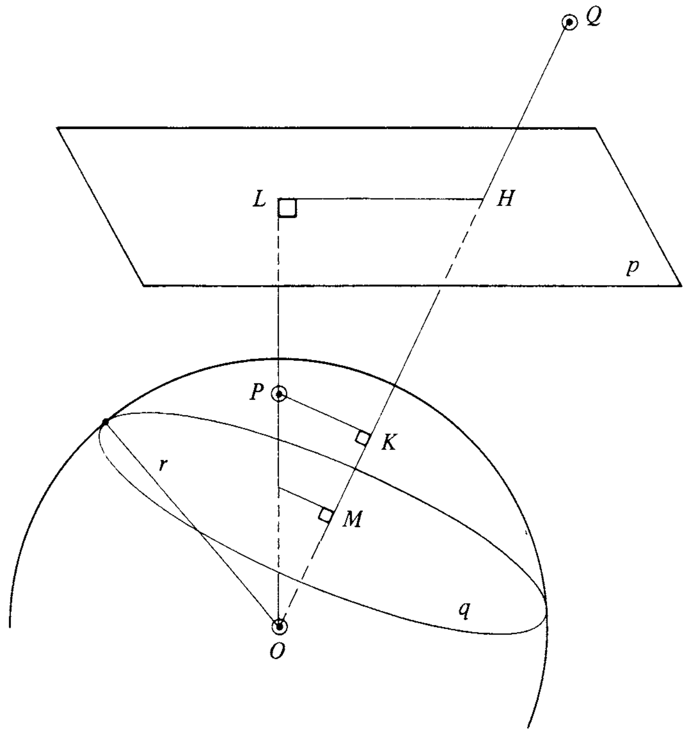


Fig. 3a–c. Polar reciprocation in solid geometry.



$$OQ \cdot OM = r^2 = OP \cdot OL = OH \cdot OK$$

$$OQ > OH \Leftrightarrow OM < OK$$

Fig. 4. Polar reciprocation in solid geometry.

8. A uniform polyhedron has all its faces either lying in the convex hull or passing between the center and some vertex of the hull. So the vertices of the dual either lie in the planes of the dual of the hull or lie outside some of its planes. Furthermore, because the vertices of the polyhedron are vertices of the hull, the planes of the dual are the planes of the dual of the hull. But this means that the dual is a stellation of the dual of the hull. Thus, all the nonconvex uniform polyhedral duals are stellations of the duals of their convex hulls.

This much from Cundy.

How facial planes are embedded in stellation patterns

With respect to the stellation process, a distinct advantage enters here because the duals have each a single stellation pattern because of the fact that a dual form is isohedral (i.e., all its faces are alike or congruent), just as the original is isogonal (i.e., all its vertices are alike). Thus, the face of the dual of any nonconvex uniform polyhedron is embedded in the stellation pattern of the dual of its convex hull.

The faces of nonconvex duals can be derived from the vertex figures of the original nonconvex forms. The polygons that appear as faces in these instances can take on some strange shapes, but the process of finding these shapes remains basically the same. The calculations can also be done and angular measures can be derived by the use of plane trigonometry. This will be specifically illustrated in the section dealing with the nonconvex duals later in this book.

A final word by way of introduction is in order here. The techniques of making the models shown in this book are the same as those used in my two previous books, and the materials used are simply white index cards or stiff paper. I strongly urge you, however, to make your own drawings wherever possible before you set out to make any model. You will learn a great deal about geometrical drawing by doing the work yourself. You are also invited to design your own templates, called *nets* here, as well as in my other books. It will give you some further challenge to work these out for yourself looking at your own drawings, using tracing paper as an aid and inspecting the photos for the interconnections of the parts.

No reference has been made to the use of color, but it is a fact that the use of colored tag (i.e., index-card stock), or, even better, metallic papers, can add greatly to the beauty of these shapes. Colors may also be introduced after a model has been constructed, when facial planes are easier to see. The use of color can add greatly to the attractiveness of these shapes, while at the same time adding to the time-consuming labor involved. But the end result is always worth the effort.

So now you are ready to begin your journey through this world of polyhedral duals. If the simpler models at the beginning of the book do not provide you with sufficient challenge, then you may skip them and go immediately to the nonconvex uniform duals. I can assure you that some of these are not too difficult, but others may well tax your ingenuity. But for all of them the mathematical aspects are always interesting, as is the richness of their interrelationships. You may also find yourself pursuing the many beckoning side paths that show up with each stellation pattern. Beautifully symmetrical shapes can thus be discovered along the way. Some of these side paths will be pointed out for you in the related commentary, without going into them, the reason being to keep the main theme true to the book's title. It is also true that stellated forms are bewilderingly numerous. A fuller treatment of stellated models could well be the topic of a future book, authored not necessarily by me, perhaps, but by anyone who finds never-ending pleasure in the discovery of new polyhedral shapes.

I. The five regular convex polyhedra and their duals

The five regular solids, also called the Platonic solids, are well known. If you have these as models to work with now, you will find that the notion of duality can very easily be illustrated with regard to them.

The tetrahedron is the simplest of all polyhedra. It has only four faces, each of which is an equilateral triangle. It has four trigonal vertices, which means that three face angles surround each vertex. Finally, it has six edges. You see immediately that an interchange in number and kind of faces and vertices leaves the number four unchanged. Because the dual of any polyhedron always keeps the same number of edges as the original from which it is derived, the number six must be kept for the number of edges. This simple description shows you that the tetrahedron is its own dual; that is, the dual of a tetrahedron is another tetrahedron.

If you look now at the octahedron, you see that it has eight faces, each of which is an equilateral triangle. It has six vertices, which can be called tetragonal, because four face angles surround each vertex. Finally, it has twelve edges. An interchange in number and kind of faces and vertices implies that its dual must have eight trigonal vertices and six tetragonal faces, and its edges must still number twelve. Another word for tetragonal is the word quadrangular, a figure or polygon with four angles. This means it also has four sides. But the question is What shape must it have? The answer is found by observing that the octahedron has a square as its vertex figure. Therefore the dual must have faces in the shape of a square. If three such squares must surround each vertex, you see that the dual of the octahedron must be the cube or hexahedron.

Now look at the cube and go through the same steps of consideration. The cube has six faces, each a perfect square. It has eight trigonal vertices and twelve edges. Its vertex figure is an equilateral triangle. So the dual of the cube must have eight equilateral triangles for faces and six tetragonal vertices, and the number of edges remains twelve. But this is a description of the octahedron. So you see that the dual of the cube is the octahedron.

Two more regular polyhedra are left to complete this introductory investigation of the five regular solids. They are the icosahedron and the dodecahedron. The icosahedron has twenty equilateral triangles for faces. It has twelve pentagonal vertices and thirty edges. An interchange in number and kind of faces and vertices, while retaining the number of edges, will lead you to see that its dual is the dodecahedron. Because the dodecahedron has twelve pentagonal faces and twenty trigonal vertices, with the number of edges still remaining at thirty, you see that its dual is the icosahedron.

This simple investigation shows that the five regular solids have duals within the same set of five. But this investigation did not take into account the polar reciprocal relationship. This can be applied mathematically, but the results are not particularly enlightening. This is so because, first of all, it is too powerful a tool to use for such simple shapes and, secondly, reciprocation in the midsphere rather than in the circumsphere gives a more interesting result. Such a reciprocation implies that the respective edges of any one of the regular solids can be made to become the perpendicular bisectors of the corresponding edges of its dual. This is a particularly attractive arrangement for model making, one that is usually shown in books about polyhedra. The tetrahedron with its dual becomes a compound of two tetrahedra, a stellated form of the octahedron, as shown in *Polyhedron models* **19**. The octahedron with its dual, the cube, is a stellated form of the cuboctahedron, as shown in *Polyhedron models* **43**. Finally, the icosahedron with its dual, the dodecahedron, is a stellated form of the icosidodecahedron, as shown in *Polyhedron models* **47**.

A different technique for model making is suggested in this book for the five regular solids. This technique can then be carried over very satisfactorily into the models of semiregular solids.

Photos 1 through 5 show each of the five regular solids embedded inside its own dual, but this dual appears only as an edge model. The geometrical and numerical details follow the elaboration given in *Spherical models* (pp. 125–31). These models are designed so that a vertex of the inner solid coincides with the incenter of a face of the dual. The faces of these duals, however, are merely suggested by the edges that lie outside the respective edges of the inner model. Thus, the edges are still perpendicular bisectors of each other, but not through each other, called skew lines; that is, the midpoint of one edge lies directly above the midpoint of the other on a radial line or a central axis of symmetry. Figures 5 through 9 show how the parts for these models may be drawn. First, a vertex part of the inner solid is laid out, but this part is shown in relation to the entire face of the inner solid. Then a drawing is given for the part needed to make an edge model of the dual. The lower-case letters on these drawings indicate where tabs are needed for cementing the parts together.

You may begin your work of making these models by first making your own templates or nets. Then cut out from card stock or stiff paper a sufficient number of parts, as many as may be needed in each instance. Next cement the vertex part into the edge design part, forming a trigonal, tetragonal, or pentagonal cup or inverted pyramid without a base holding a vertex part inside it. Finally, cement all these cups or inverted pyramids together, the lateral face of one to the lateral face of another, until the model is complete.

You may, of course, alter the design of any one of these, or all of them, to show other

Photo 1. Tetrahedron (**1**).

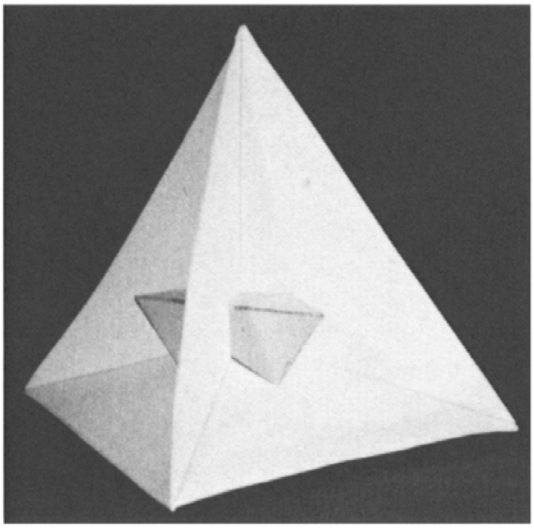
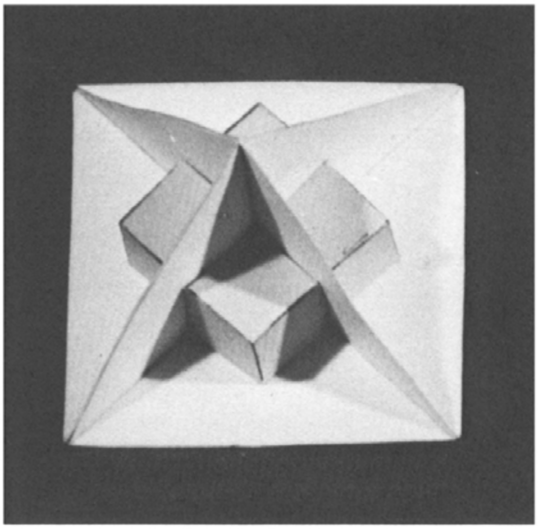


Photo 2. Cube or hexahedron (**2**).



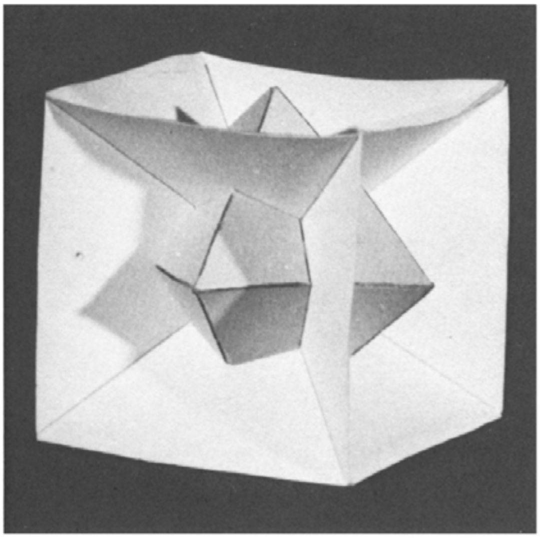


Photo 3. Octahedron (**3**).

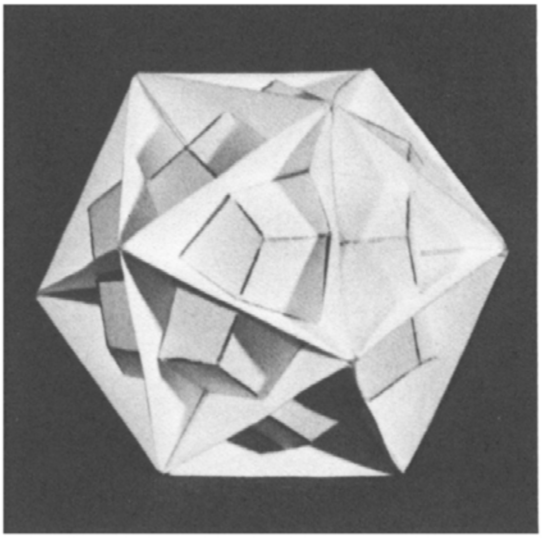
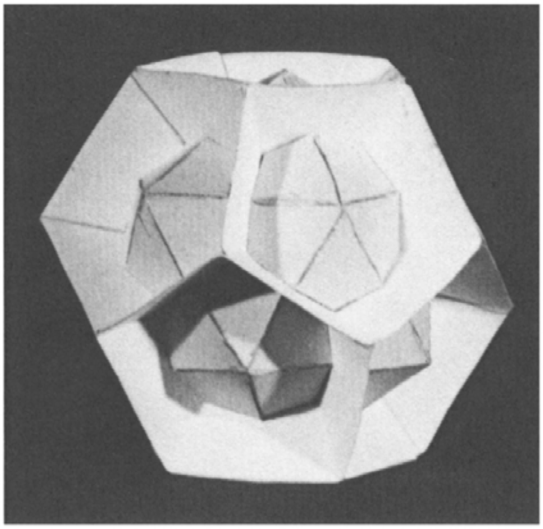


Photo 4. Dodecahedron (**4**).

relationships with respect to their size, but not their shape. For example, the inner polyhedron can be made very small in relation to its dual in a beginning model. Then several other models can be made showing successive stages in the growth of this inner polyhedron until it reaches the position suggested by the photos. Continuing growth can then be shown with more models until the edge model disappears inside the solid model. This happens when the vertices of the edge model coincide with the incenters of the faces of the solid model. A set of such models would suggest continuous transformations of these polyhedral shapes. Such continuous transformations are a rich source from which many polyhedral relationships are derived.

Although the five regular solids are very simple shapes in themselves, within them there lies hidden the whole world of polyhedral symmetry. Like musical variations on a theme, the five regular solids reveal their presence in countless ways in all the more complex shapes that will appear later on in this book. You will see that this is already true in the next set of uniform polyhedra to be considered (i.e., the thirteen semiregular solids and their duals).

Photo 5. Icosahedron (**5**).



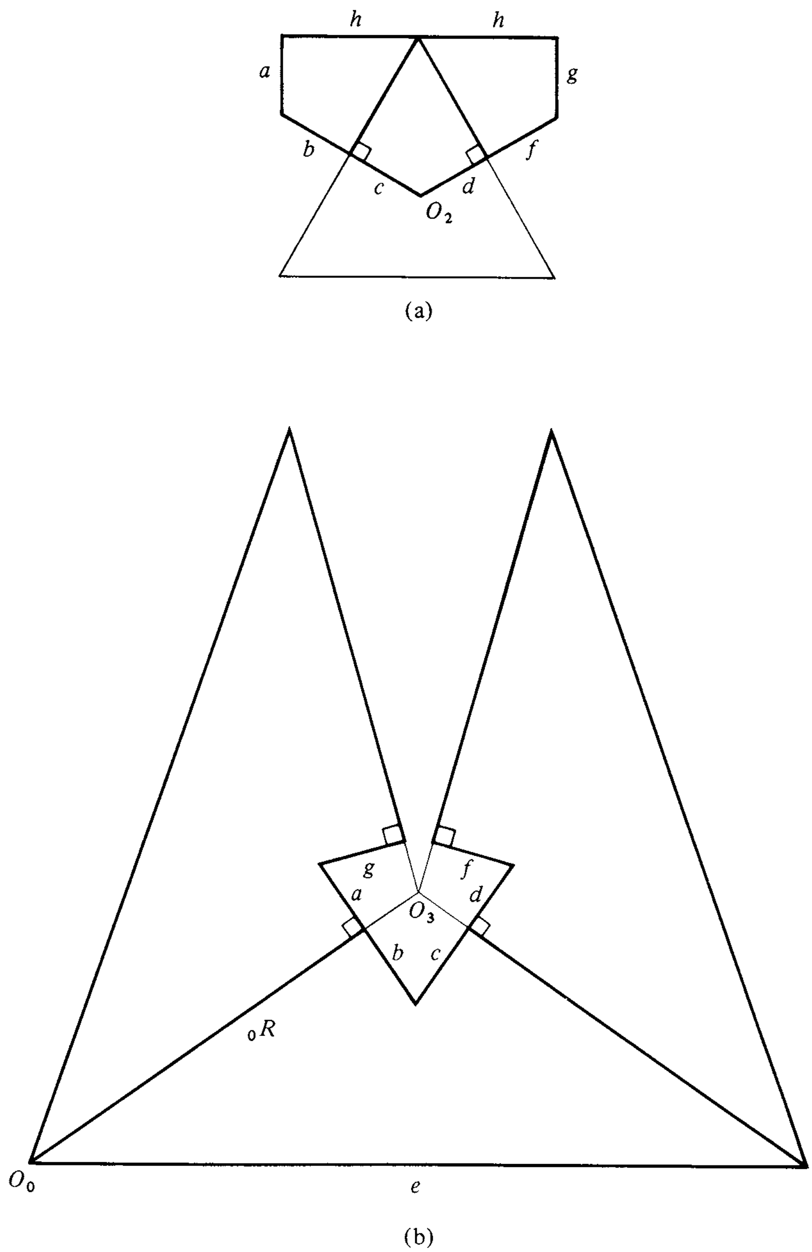


Fig. 5. Patterns for the dual of the tetrahedron (1).