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Ordered Permutation Groups

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To my wife, Rona,
and my mother, Vicky;

and in memory of
my father, Jack (1912-1969)
and my aunt, Ethel (1902-1980).

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PREFACE

In the past thirty years, groups of order-preserving permutations of totally ordered sets have been extensively studied. The purpose of these notes is to provide a uniform, systematic account of this research and its applications. In the first half of this book (Parts I and II), I attempt a streamlined (and I trust intuitive) presentation of the main results in the structure theory, taking full advantage of recent research. In Chapters 3 and 5, the study of such permutation groups is reduced to an investigation of the basic building blocks of the subject, "primitive" order-preserving permutation groups. These are classified and examined in Chapters 2 and 4. The second half of the book is devoted to various applications of the structure theory; e.g., to the construction of infinite simple groups. Most of the chapters in it can be read quite independently of each other. I have chosen the topics to illustrate the use of the structure theory in a wide variety of settings, but readily admit that the selection is strongly influenced by my own quirks and prejudices. The total order on a set naturally lifts to a lattice order on its group of order-preserving permutations. Since every lattice-ordered group can be embedded (as a group and lattice simultaneously) in such a lattice-ordered group--the Cayley-Holland Theorem, see Appendix I--the theory can be used to study lattice-ordered groups. For example, I will show: every lattice-ordered group can be embedded in a simple divisible lattice-ordered group in which any two elements greater than the identity are conjugate; and my favourite application: there is a finitely presented lattice-ordered group with insoluble word problem (Whereas analogues of both these results for groups can be proved without recourse to permutation groups, I know of no proof for lattice-ordered groups which avoids using groups of order-preserving permutations of totally ordered sets).

My main aim throughout has been to impart the intuition and excitement of the subject. If any of this comes across, I will be most satisfied.

With the exception of Chapter 1, I have begun each chapter with an account of the goals and main theorems it contains (though often not stated in the generality in which they are proved). I hope that this will help the reader to get a good overall picture of the subject, as well as whet his or her appetite; it should also make it easier to locate results in the book. I have included several theorems which have not previously been published, so these notes hold something new even for the specialist. Since anyone who has read Parts I and II and a few of the applications is at the forefront of the subject, I have included a list of unsolved problems which I think are of especial interest. I am sure that the reader will find others to work on. Caution: in order to give as uncluttered and intuitive an account as possible, I have often put more stringent hypotheses on theorems than are really necessary. This eliminates some technicalities. The interested reader should consult the literature to find out the full story (an annotated bibliography is included to make this easier).

As a result of my earlier book of the same title (published in 1976 by Bowling Green State University and now out of print), David Tranah approached me to write a monograph on the subject for Cambridge University Press. These notes are the consequence. Unlike the 1976 version, they are not encyclopaedic but are designed instead for a much broader audience. I have confined my attention to those theorems in the structure theory which have been most fruitful for applications, and provided simpler proofs to some when more recent research has made this possible. Also, many of the applications given here are the result of work done in the last five years, and so did not appear in the 1976 version.

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This book was obtained by photocopying the typed version of the manuscript. Consequently, everyone reading it will appreciate the excellence of the typist, Linda Shellenbarger, and the deep gratitude I owe her. I am also beholden to Bruce Lyle for his superb illustrations; Ashok Kumar Arora and Manfred Droste for reading the handwritten version and making suggestions for needed improvements; Todd Feil for reading the typed version; and Rona Glass for the index.

When it comes to thanking mathematicians, I do not know where to begin. I have had the great fortune to find, without exception, only kindness and generous assistance from teachers, lecturers, professors and colleagues on both sides of the Atlantic. Since a list of names would be far too long, let me just humbly thank all I have come in contact with. However, I would especially like to thank W. Charles Holland who, first as a thesis director and then as a colleague, has so willingly shared many insights and speculations with me. Stephen McCleary has also been a constant source of encouragement and contagious enthusiasm. In addition, I would like to thank my colleagues in the mathematics department here for their friendliness and interest in my work. Like everyone in the States, they have made my stay here most enjoyable. It gives me great pleasure to acknowledge my thanks to them in print.

Finally, I would like to thank David Tranah at Cambridge University Press for his encouragement, help, patience and good humour throughout the ordeal of writing these notes. It is quite accurate to say that without his urging, this book would never have been written. I trust it will not be held against him for too long!

Bowling Green, Ohio, U.S.A
Whitmonday, 1981

BACKGROUND TERMS AND NOTATION

1. *SET THEORY*.

Let A and B be sets. We will write $A \subseteq B$ if A is a subset of B, and $A \subsetneq B$ if A is a proper subset of B. We will use $A \backslash B$ for $\{a \in A: a \notin B\}$. If f is a function from A into B, let $Af = \{af: a \in A\}$, the *range* (or *image*) of f; and if $C \subseteq A$, let $f|C = \{(a,b) \in f: a \in C\}$, the *restriction* of f to C. We will use $\emptyset$ for the empty set, ω for the least infinite ordinal, ω_1 for the least uncountable infinite ordinal, etc. So $\omega = \{0,1,2,...\}$. We will often identify ω with $\aleph_0$, ω_1 with $\aleph_1$, etc. We will write $|A|$ for the cardinality of A, and say A is *countable* if $|A| = \aleph_0$.

2. *PARTIALLY ORDERED SETS*.

A set Ω with a reflexive, antisymmetric, transitive relation $\leq$ defined on it is called a *partially ordered set*, or *p.o. set*. As usual, we will write $\alpha < \beta$ for $\alpha \leq \beta$ & $\alpha \neq \beta$. A p.o. set in which $\alpha \leq \beta$ only if $\alpha = \beta$ is said to be *trivially ordered*. In contrast, a p.o. set Ω is said to be *totally ordered* or *linearly ordered* (or a *chain*, for short) if for all $\alpha,\beta \in \Omega$, $\alpha \leq \beta$ or $\beta \leq \alpha$. A p.o. set Ω in which each $\alpha,\beta \in \Omega$ have a least upper bound (denoted by $\alpha \vee \beta$) and a greatest lower bound (denoted by $\alpha \wedge \beta$) is called a *lattice*. If Ω is a lattice and $\alpha \vee (\beta \wedge \gamma) = (\alpha \vee \beta) \wedge (\alpha \vee \gamma)$ for all $\alpha,\beta,\gamma \in \Omega$, then Ω is said to be a *distributive lattice*; equivalently, if $\alpha \wedge (\beta \vee \gamma) = (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$ for all $\alpha,\beta,\gamma \in \Omega$. If Ω is a p.o. set and $T \subseteq \Omega$, we say that T is *dense* in Ω if whenever $\alpha < \beta$ in Ω, there is $\tau \in T$ such that $\alpha < \tau < \beta$ (cf., dense in itself, page 83). We will call Ω

dense if it is dense in Ω. If $\alpha < \beta$ in a p.o. set Ω, we write $[\alpha,\beta] = \{\gamma \in \Omega: \alpha \leq \gamma \leq \beta\}$ for the closed interval, and $(\alpha,\beta) = \{\gamma \in \Omega: \alpha < \gamma < \beta\}$ for the open interval. If $\gamma \in [\alpha,\beta]$, we say that γ lies *between* α and β; if $\gamma \in (\alpha,\beta)$, that γ lies *strictly between* α and β. A subset T of a p.o. set Ω is said to be an *interval* of Ω (or *convex* in Ω) if whenever $\tau_1,\tau_2 \in T$ and $\gamma \in \Omega$ lies between τ_1 and τ_2, then $\gamma \in T$. If Ω is a p.o. set, Ω^* is the set Ω with the reverse ordering; i.e., $\alpha \leq \beta$ in Ω^* precisely when $\alpha \geq \beta$ in Ω. So Ω^* is also a p.o. set. If I is a chain and $\{\Omega_i: i \in I\}$ is a family of p.o. sets, then $\overleftarrow{\bigcup}\{\Omega_i: i \in I\}$ is the set $\bigcup\{\Omega_i: i \in I\}$ ordered by: $\alpha \leq \beta$ if $\alpha \in \Omega_i$, $\beta \in \Omega_j$ and $i < j$ or $(i = j \;\&\; \alpha \leq \beta$ in $\Omega_i)$; $\overrightarrow{\bigcup}\{\Omega_i: i \in I\}$ is the same set but ordered by: $\alpha \leq \beta$ if $\alpha \in \Omega_i$, $\beta \in \Omega_j$ and $i > j$ or $(i = j \;\&\; \alpha \leq \beta$ in $\Omega_i)$.

We will use $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ for the chains of integers, rationals and reals respectively (under the natural order). $\mathbb{Z}^+$, $\mathbb{Q}^+$ and $\mathbb{R}^+$ will denote the sets of strictly positive integers, rationals and reals, respectively.

If Ω_1 and Ω_2 are lattices, then a *lattice isomorphism* is a one-to-one map of Ω_1 onto Ω_2 that preserves the lattice operations. In the special case that Ω_1 and Ω_2 are chains, lattice isomorphisms are called *ordermorphisms*, and the chains are said to be ordermorphic.

3. *GROUP THEORY*.

If G is a group and H is a subgroup of G, we will write $H \triangleleft G$ if H is normal. We will write $H \rtimes G$ for a group with normal subgroup H, having quotient by H isomorphic to G; i.e., an extension of H by G. If C is a subgroup of a group G, $R(C) = \{Cg: g \in G\}$, the set of *right cosets* of C in G. All isomorphisms are onto. $\mathbb{Z}_2$ will denote the two element group. We will use *e* for the identity element of any group.

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4. *MODEL THEORY*.

Our language will be the first order language of groups, lattices with e, or ℓ-groups. So our formulae will be built up in the usual way from equality of group words, lattice words or ℓ-group words by using ¬ (not), & or $\bigwedge$ (and), or or $\bigvee$ (or), ∀ (for all) and ∃ (there exists). Note (∃ integer n) is not permitted; nor are infinite conjunctions, etc. A formula without free variables is called a *sentence*. We will write $G \models \theta$ if the sentence θ holds in the group (lattice, ℓ-group) G, and $G \equiv H$ if the groups (lattices with e, ℓ-groups) satisfy the same sentences of the language of groups (lattices with e, ℓ-groups).

5. *CONVENTIONS ADOPTED*.

Capital Greek letters are used for chains, small Greek letters for elements of them (exception: α, κ, λ, μ, ν, ξ, η, ζ are sometimes used in other contexts, e.g., as ordinals, members of index sets; ψ, ϕ, θ are usually reserved for mappings). C, G, H will denote groups, lattices with e, or ℓ-groups and c, f, g, h, k, x, y, z will denote elements of them. I and J are reserved for general index sets; i, j for elements of these index sets. Script letters are used for equivalence relations; *k* is reserved for the set of covering pairs of congruences (see Sections 1.6, 1.7 and Chapter 3), and K for an element of *k*.

We will sometimes use // to denote the end of a proof; sometimes Q.E.D.; sometimes nothing.

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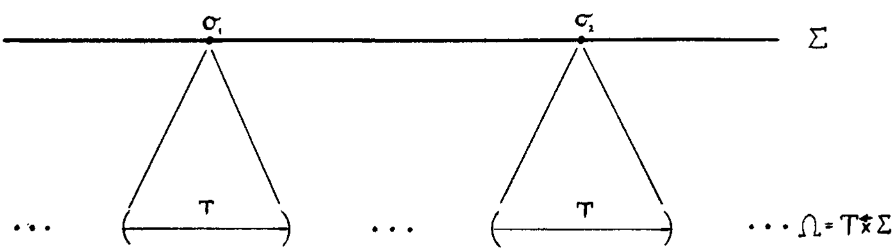
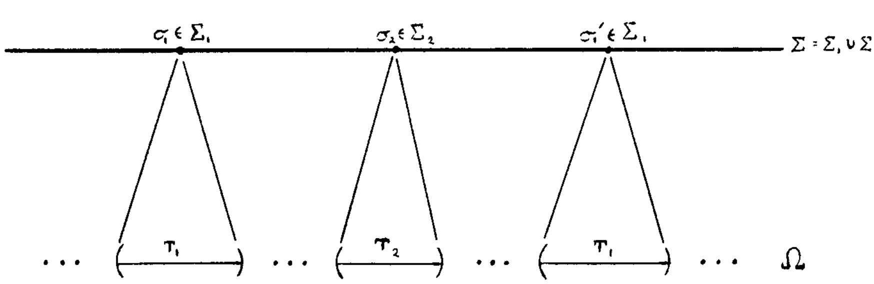
EXPLANATION OF DIAGRAMS.

The picture



will be used to illustrate the chain Ω obtained from Σ by replacing each $\sigma \in \Sigma$ by a copy of the chain T; i.e., $\Omega = \underline{T \overleftarrow{\times} \Sigma} = \{(\tau,\sigma): \tau \in T, \sigma \in \Sigma\}$ ordered by: $(\tau_1,\sigma_1) < (\tau_2,\sigma_2)$ if $\sigma_1 < \sigma_2$ or $(\sigma_1 = \sigma_2 \ \& \ \tau_1 < \tau_2)$. Let Σ, T_1, T_2 be chains, with Σ_1, Σ_2 a partition of Σ.

The picture



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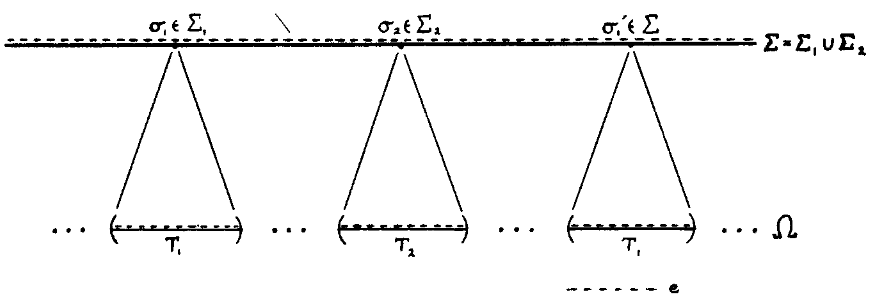
will be used to illustrate the chain Ω obtained by replacing each $\sigma_i \in \Sigma_i$ by a copy of T_i $(i = 1,2)$. That is,
$\Omega = \{(\tau_1,\sigma_1): \tau_1 \in T_1,\ \sigma_1 \in \Sigma_1\} \cup \{(\tau_2,\sigma_2): \tau_2 \in T_2,\ \sigma_2 \in \Sigma_2\}$
ordered by: $(\tau,\sigma) < (\tau',\sigma')$ if $\sigma < \sigma'$ (in Σ) or $(\sigma = \sigma'$ & $\tau < \tau')$.

For such layered diagrams, we will use the horizontal (rather than the diagonal) to depict the identity map; e.g.,



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PART I

OPENING THE INNINGS