CHAPTER 1
INTRODUCTION

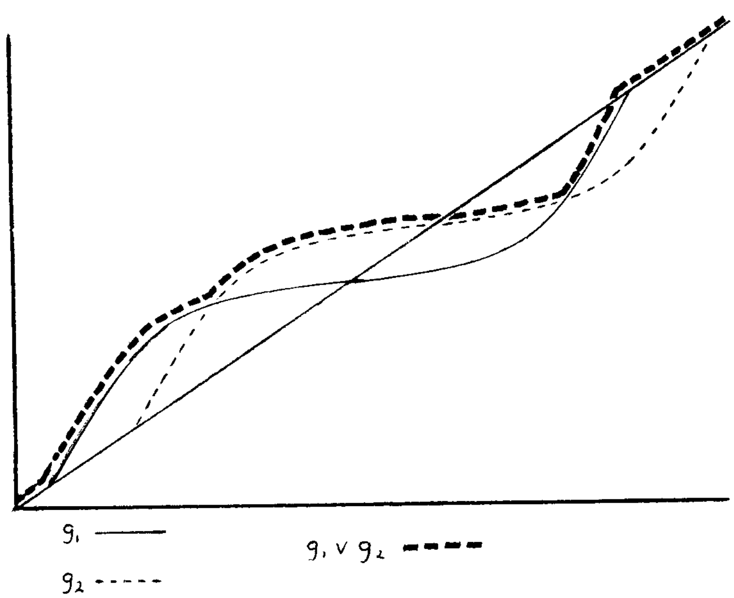
The theory of permutation groups (especially in the finite case) has been developed over a long period of time. One motivation for the study was geometric. Another was provided by Cayley's Theorem that every group is (isomorphic to) a subgroup of a permutation group. More recently, infinite permutation groups have been studied and used to provide examples (and counterexamples) in infinite group theory.

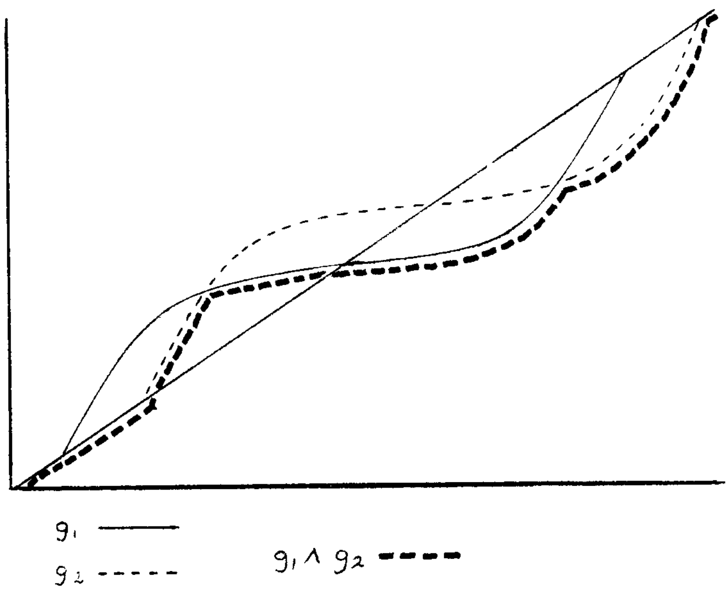
On the other hand, ordered permutation groups have been studied for a relatively short time. Again, one motivation to examine them was geometric. Another was provided in 1963 by W. Charles Holland's analogue of Cayley's Theorem: Every lattice-ordered group is (isomorphic to) a subgroup of the lattice-ordered group of all order-preserving permutations of a chain (linearly ordered set). A further reason was provided by model theory: If T is a first order theory having infinite models and $\langle\Omega,\leq\rangle$ is a chain, then there is a model $\mathfrak{A}$ of T containing Ω such that each order-preserving permutation of Ω extends to an automorphism of $\mathfrak{A}$ (Ehrenfeucht and Mostowski [56]). Hence any study of the group of automorphisms of models of such theories should begin with a study of groups of order-preserving permutations of chains.

In this chapter we will devote ourselves to basic definitions and results. Many of these are routine translations of theorems concerning permutation groups to the ordered permutation group setting. The presentation we give is self-contained and does not require prior knowledge of the theory of permutation groups. However, anyone wishing to become conversant with the roots of the subject could do worse than refer to H. Wielandt's works [64], [67] and [69]. In contrast, some of the theory we present here (especially in Sections 1.8, 1.10 and 1.11) is peculiar to the ordered case, though we have included it in this chapter only if it is rather straightforward.

1.1. $A(\Omega)$.

Let Ω be a chain. We define $A(\Omega)$ to be the set of all order-preserving permutations of Ω; i.e., $g \in A(\Omega)$ if and only if g is a one-to-one function from Ω onto Ω that satisfies: if $\alpha, \beta \in \Omega$ and $\alpha < \beta$, then $\alpha g < \beta g$. $A(\Omega)$ is a group under composition: if $f, g \in A(\Omega)$ define fg by: $\alpha(fg) = (\alpha f)g$ $(\alpha \in \Omega)$. Further, we can define an order $\leq$ on $A(\Omega)$ via: $f \leq g$ if $\alpha f \leq \alpha g$ for all $\alpha \in \Omega$. This order on $A(\Omega)$ is called the *pointwise order*. It is immediate that if $f,g,h \in A(\Omega)$ and $f \leq g$, then $fh \leq gh$ and $hf \leq hg$. Moreover, any pair of elements $g_1, g_2 \in A(\Omega)$ has a supremum (least upper bound) and infimum (greatest lower bound) in $A(\Omega)$--which we denote by $g_1 \vee g_2$ and $g_1 \wedge g_2$ respectively--namely:
$\alpha(g_1 \vee g_2) = \max\{\alpha g_1, \alpha g_2\}$ and $\alpha(g_1 \wedge g_2) = \min\{\alpha g_1, \alpha g_2\}$ $(\alpha \in \Omega)$.
(Since Ω is a chain, $\alpha g_1 \leq \alpha g_2$ or $\alpha g_2 \leq \alpha g_1$.)





EXAMPLE 1.1.1. Let $\Omega = \mathbb{Z}$, the chain of integers (in the natural ordering). The order-preserving permutations of $\mathbb{Z}$ are just translations by integers. If $t_n \in A(\mathbb{Z})$ denotes translation by n, then the map $n \mapsto t_n$ is a group isomorphism of $\mathbb{Z}$ onto $A(\mathbb{Z})$ that preserves order; i.e., if $n \leq m$, then $t_n \leq t_m$ in the pointwise ordering ($\alpha t_n = \alpha + n \leq \alpha + m = \alpha t_m$ for all $\alpha \in \mathbb{Z}$). In this example, for each $\alpha, \beta \in \Omega$, there is a unique $g \in A(\Omega)$ such that $\alpha g = \beta$, and the pointwise order is a total order.

EXAMPLE 1.1.2. Let $\Omega = \mathbb{R}$, the chain of real numbers (in the natural ordering). Let $g_0 \colon \alpha \mapsto 2\alpha$ ($\alpha \in \mathbb{R}$). Then $g_0 \in A(\mathbb{R})$. However, $1g_0 = 2$ (so $e \not\geq g_0$) and $(-1)g_0 = -2$ (so $e \not\leq g_0$). Hence the pointwise order is not a total order; so there is no ordermorphism of $\mathbb{R}$ onto $A(\mathbb{R})$. Let $f_0 \colon \alpha \mapsto \alpha + 1$ ($\alpha \in \mathbb{R}$). Then $f_0 \in A(\mathbb{R})$ and $f_0 g_0 \neq g_0 f_0$. Thus $A(\mathbb{R})$ is not an Abelian group. If $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{R}$ with $\alpha_1 < \beta_1$ & $\alpha_2 < \beta_2$, there is $h_0 \in A(\mathbb{R})$ such that $\alpha_1 h_0 = \alpha_2$ & $\beta_1 h_0 = \beta_2$; e.g., h_0 is the straight line passing through (α_1, α_2) and (β_1, β_2).

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EXAMPLE 1.1.3. Let $\mathbb{Z}_i$ $(i = 1,2)$ be disjoint copies of $\mathbb{Z}$ and let $\Omega = \mathbb{Z}_1 \overset{\leftarrow}{\cup} \mathbb{Z}_2$. Any two points of $\mathbb{Z}_2$ have only a finite

number of elements of Ω between them. Hence if $g \in A(\Omega)$, the images under g of any two points of $\mathbb{Z}_2$ still enjoy this property. Therefore $\mathbb{Z}_2 g = \mathbb{Z}_1$ or $\mathbb{Z}_2$. If $\alpha_i \in \mathbb{Z}_i$ $(i = 1,2)$, then $\alpha_1 < \alpha_2$; so $\alpha_1 g < \alpha_2 g$. It follows that $\mathbb{Z}_i g = \mathbb{Z}_i$ $(i = 1,2)$. Consequently, by Example 1.1.1, there is an order-preserving group isomorphism between $A(\Omega)$ and $\mathbb{Z} \oplus \mathbb{Z}$ if we order $\mathbb{Z} \oplus \mathbb{Z}$ by: $(m_1,n_1) \leq (m_2,n_2)$ if $m_1 \leq m_2$ & $n_1 \leq n_2$ (in $\mathbb{Z}$).

1.2. *ACTIONS OF GROUPS ON CHAINS.*

Let Ω be a chain and G a group. An *action* of G on Ω is a triple (G,Ω,θ) where $\theta\colon G \to A(\Omega)$ is a homomorphism. The *kernel* of θ, *ker(θ)*, is called the *lazy subgroup* (if $g \in \ker(\theta)$, $\alpha(g\theta) = \alpha$ for all $\alpha \in \Omega$). If θ is one-to-one (i.e., the lazy subgroup is $\{e\}$), we call the action *faithful* and say that (G,Ω,θ) is a *representation* of G. It is clear that if (G,Ω,θ) is an action, $(G/\ker(\theta),\Omega,\theta^*)$ is a faithful action where $(\ker(\theta)g)\theta^* = g\theta$ $(g \in G)$.

EXAMPLE 1.2.1. Consider $(\mathbb{R},\mathbb{R},\theta)$ where $\alpha(r\theta) = \alpha + r$ $(\alpha, r \in \mathbb{R})$. Then $(\mathbb{R},\mathbb{R},\theta)$ is a faithful action, and so a representation of $\mathbb{R}$. Observe that the group $\mathbb{R}\theta$ (the group operation is composition) is isomorphic to the additive group of reals. Hence $\mathbb{R}\theta \neq A(\mathbb{R})$. Indeed, the pointwise ordering on $\mathbb{R}\theta \subseteq A(\mathbb{R})$ is total (if $r,s \in \mathbb{R}$, $r\theta \leq s\theta$ or $s\theta \leq r\theta$ where $\leq$ is the pointwise ordering).

LEMMA 1.2.2. *If* (G,Ω,θ) *is an action of* G *on a chain* Ω, *then the map* $(\alpha,g) \mapsto \alpha(g\theta)$ *of* $\Omega \times G$ *onto* Ω *satisfies* *(i)* $(\alpha,e) \mapsto \alpha$, *(ii) the image of* (α,g) *is less than the image of* (β,g) *whenever* $\alpha < \beta$, *and (iii)* (α,fg) *has the same image as* (β,g) *where* β *is the image of* (α,f). *Conversely, any map from* $\Omega \times G$ *into* Ω *satisfying (i), (ii) and (iii) gives rise to a homomorphism of* G *into* $A(\Omega)$ *(and hence to an action of* G *on* Ω*).*

Proof: That any action gives rise to a map satisfying (i), (ii) and (iii) is utterly obvious. Conversely, define θ by: $\alpha(g\theta)$ is the image of (α,g) $(\alpha \in \Omega,\ g \in G)$. By (i), $(\alpha,g^{-1}g) \mapsto \alpha$ so by (iii) $\beta(g\theta) = \alpha$ where $\beta = \alpha(g^{-1}\theta)$. Hence, by (ii), $g\theta \in A(\Omega)$. Thus $\theta\colon G \to A(\Omega)$. By (iii), θ is a homomorphism.

We will often write (G,Ω) instead of (G,Ω,θ) when θ is clear from context. In particular, if $G \subseteq A(\Omega)$, θ will be assumed to be the identity (unless specifically stated to the contrary). In this case we will endow G with the pointwise ordering and say that (G,Ω) is an *ordered permutation group*. So an ordered permutation group (G,Ω) is a subgroup of $A(\Omega)$ together with its inherited action on Ω. If (G,Ω) is an ordered permutation group and for each $g_1,g_2 \in G$, the elements $g_1 \vee g_2$ and $g_1 \wedge g_2$ of $A(\Omega)$ actually belong to G, we will say that (G,Ω) is an *ℓ-permutation group*. So for any chain Ω, $(A(\Omega),\Omega)$ is an ℓ-permutation group.

EXAMPLE 1.2.3. Let $\Omega = \mathbb{R}$ and G be the group of order-preserving permutations of $\mathbb{R}$ that are finitely piecewise linear. That is, $g \in G$ if $g \in A(\mathbb{R})$ and there are $\alpha_0 < \ldots < \alpha_n$ in $\mathbb{R}$ such that $g|(-\infty,\alpha_0]$, $g|[\alpha_i,\alpha_{i+1}]$ and $g|[\alpha_n,\infty]$ are linear $(0 \le i < n)$. Then $(G,\mathbb{R})$ is an ℓ-permutation group. Note G is not Abelian $(f_0,g_0 \in G$ where f_0,g_0 are defined in Example 1.1.2).

If (G,Ω) is an action, we may think of Ω as an algebra with unary operations g $(g \in G)$, and binary operations $\vee$ and $\wedge$ given by $\vee(\alpha,\beta) = \max\{\alpha,\beta\}$ and $\wedge(\alpha,\beta) = \min\{\alpha,\beta\}$. A *subalgebra* (or *G-invariant set*) will then be a subset Δ of Ω such that $\Delta g = \Delta$ for all $g \in G$. Note that $\emptyset$ is a G-invariant set. If $\alpha \in \Omega$, the subalgebra generated by α is denoted by αG and is called the *G-orbit of* α. So $\alpha G = \{\alpha g: g \in G\}$.

THEOREM 1.2.4. *Let* (G,Ω) *be an action. Every G-invariant set is the union of the G-orbits it contains. Moreover, the set of G-orbits of points of* Ω *partitions* Ω. *Thus the collection of G-invariant sets forms a complete Boolean algebra under union and intersection.*

Proof: If $\beta \in \alpha G$, then $\beta = \alpha g$ for some $g \in G$. Hence $\beta G = \alpha g G = \alpha G$. So the G-orbits partition Ω. The rest of the theorem is now obvious ($\emptyset$ and Ω are the 0 and 1 of the complete Boolean algebra).

If $\Omega = \alpha G$ for some (and hence every) $\alpha \in \Omega$, then the action of G on Ω is said to be *transitive*. Examples 1.1.1, 1.1.2, 1.2.1 and 1.2.3 are transitive ℓ-permutation groups but Example 1.1.3 shows that $(A(\Omega),\Omega)$ need not be transitive. If $(A(\Omega),\Omega)$ is transitive, we will simply say that Ω is *homogeneous*. (Caution: This is not the model theory usage of the word; homogeneous for us is "1-homogeneous" of model theory.)

Throughout the book we will assume that

IF (G,Ω) IS TRANSITIVE, THEN Ω IS INFINITE.

This eliminates the trivial case that $G = \{e\}$ and $|\Omega| = 1$.

1.3. *PARTIALLY ORDERED GROUPS*.

A group G together with a partial order $\leq$ on the elements of G is called a *p.o.group* (*partially ordered group*) if whenever $f,g,h \in G$, $f \leq g$ implies $fh \leq gh$ and $hf \leq hg$. If, in addition, $\leq$ is a lattice (i.e., $g_1 \vee g_2$ (the supremum or least upper bound of g_1 and g_2) and $g_1 \wedge g_2$ (the infimum or greatest lower bound of g_1 and g_2) exist for all $g_1, g_2 \in G$), we say that G is a *lattice-ordered group*, or *ℓ-group* for short. Note that if G is an ℓ-group, $(f \vee g)h = fh \vee gh$, $h(f \vee g) = hf \vee hg$ and dually for $\wedge$; also $f \vee g = (f^{-1} \wedge g^{-1})^{-1}$ and $f \wedge g = (f^{-1} \vee g^{-1})^{-1}$. If G is a p.o. group in which $\leq$ is a total order, we say that G is a *totally ordered group*, or *o-group* for short. If G is a p.o. group and $e \leq g \in G$, we say that g is *positive*; if $e < g \in G$, we say that g is *strictly positive*.

As examples, $\mathbb{Z}$ and $\mathbb{R}$ with the usual orders are o-groups, and G is an ℓ-group under the pointwise ordering whenever (G,Ω) is an ℓ-permutation group. In particular, $A(\Omega)$ is an ℓ-group for every chain Ω. Another example of an ℓ-group is $\mathbb{Z} \oplus \mathbb{Z}$ ordered as in Example 1.1.3. If (G,Ω) is an ordered permutation group, then G is a p.o. group.

If G is an ℓ-group and H is a subgroup of G that is closed under the lattice operations ($\vee$ and $\wedge$), then H is said to be an *ℓ-subgroup* of G. Hence an ordered permutation group (H,Ω) is an ℓ-permutation group precisely when H is an ℓ-subgroup of $A(\Omega)$.

An *o-homomorphism* from a p.o. group G into a p.o. group H is a group homomorphism from G into H which preserves the partial ordering. If G and H are ℓ-groups and the o-homomorphism preserves the lattice operations, we will call the o-homomorphism an *ℓ-homomorphism*. If ψ is one-to-one and both ψ and ψ^{-1} are o-homomorphisms, we say that ψ is an *o-embedding*. If ψ is a one-to-one ℓ-homomorphism, we say that ψ is an *ℓ-embedding*. It is trivial to show that any ℓ-embedding is an o-embedding. An (ℓ-)o-embedding of G onto H is called an *(ℓ-)o-isomorphism* between G and H. We will frequently use the following fact (which is easy to verify): A homomorphism $\psi: G \to H$ is an ℓ-homomorphism if $(g \vee e)\psi = g\psi \vee e$ for all $g \in G$. In Example 1.1.3, $A(\Omega)$ and $\mathbb{Z} \oplus \mathbb{Z}$ are ℓ-isomorphic.

Let (G,Ω) and (H,T) be ordered (ℓ-)permutation groups. If $\phi: \Omega \to T$ is an order-preserving map and $\psi: G \to H$ is an o-(ℓ-)homomorphism, then (ψ,ϕ) is said to be an *o-(ℓ-)homomorphism* of (G,Ω) into (H,T) if $(\alpha\phi)(g\psi) = (\alpha g)\phi$ $(\alpha \in \Omega,\ g \in G)$. If, in addition, ϕ is one-to-one and ψ is an o-(ℓ-)embedding we obtain an *ordered (ℓ-)permutation group embedding* of (G,Ω) into (H,T), and an *o-(ℓ-)isomorphism* if ϕ and ψ are also onto. If $H = G$ and ψ is the identity, then ϕ is said to be an *o-(ℓ-)homomorphism* if (ψ,ϕ) is.

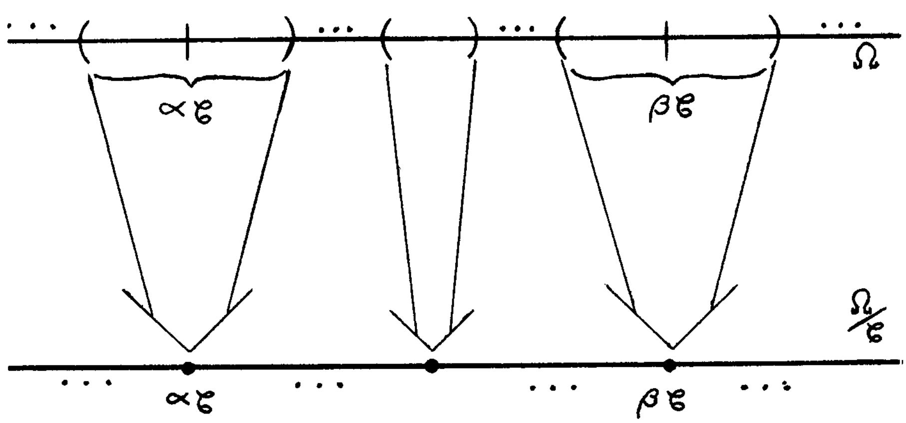
1.4. CONGRUENCES.

Let X be a partially ordered set and $Y \subseteq X$. Y is convex (in X) if $y_1, y_2 \in Y$ and $y_1 \leq x \leq y_2$ imply $x \in Y$ (see p. vii).

Let (G,Ω) be an action. A *congruence* of (G,Ω) is an equivalence relation $\mathcal{C}$ on Ω such that each equivalence class is convex and $\alpha g\,\mathcal{C}\,\beta g$ whenever $\alpha\,\mathcal{C}\,\beta$ $(\alpha,\beta \in \Omega,\ g \in G)$.

If $\mathcal{C}$ is an equivalence relation on Ω, we will write $\alpha\mathcal{C}$ for $\{\beta \in \Omega : \alpha\,\mathcal{C}\,\beta\}$, the equivalence class of $\mathcal{C}$ which contains α, and $\Omega/\mathcal{C}$ for $\{\alpha\mathcal{C} : \alpha \in \Omega\}$.

If (G,Ω) is an action with congruence $\mathcal{C}$, then we can linearly order $\Omega/\mathcal{C}$ by: $\alpha\mathcal{C} < \beta\mathcal{C}$ if $\sigma < \tau$ for all $\sigma \in \alpha\mathcal{C}$ and $\tau \in \beta\mathcal{C}$.



There is a natural action $(G,\Omega/\mathcal{C})$ given by: $(\alpha\mathcal{C})g = (\alpha g)\mathcal{C}$ $(\alpha \in \Omega,\ g \in G)$, and $\phi: \Omega \to \Omega/\mathcal{C}$ is an o-(ℓ-)homomorphism if (G,Ω) is an ordered (ℓ-)permutation group. Moreover, if L is the lazy subgroup of the action $(G,\Omega/\mathcal{C})$, then $(G/L,\Omega/\mathcal{C})$ is an ordered (ℓ-)permutation group and the natural map of (G,Ω) onto $(G/L,\Omega/\mathcal{C})$ is an ordered (ℓ-)permutation group homomorphism.

If (G,Ω) and (G,T) are ordered permutation groups and $\phi: \Omega \to T$ is an o-homomorphism, the kernel of ϕ is the equivalence relation $\mathcal{K}$ defined by: $\alpha \mathcal{K} \beta$ if $\alpha\phi = \beta\phi$. Clearly, $\mathcal{K}$ is a congruence of (G,Ω). So if $T = \Omega/\mathcal{C}$ and ϕ is the natural map of Ω onto $\Omega/\mathcal{C}$, the kernel of ϕ is $\mathcal{C}$.

The following two theorems are straightforward--either by direct proof or universal algebra.

THEOREM 1.4.1 *(Homomorphism Theorem). Let (G,Ω) be an ordered permutation group.*

(i) The kernel of an o-homomorphism is a congruence of (G,Ω).

(ii) Let $\phi: \Omega \to T$ be an o-homomorphism with kernel $\mathcal{K}$ and $\eta: \Omega \to \Omega/\mathcal{K}$ the natural map. Then there is an o-embedding $\phi': \Omega/\mathcal{K} \to T$ such that $\eta\phi' = \phi$.

(iii) If ϕ is as in (ii), then $\Omega\phi$ is a G-invariant subset of T; and if ϕ is onto, ϕ' is an ordermorphism.

THEOREM 1.4.2 *(Correspondence Theorem). Let (G,Ω) be an ordered permutation group and $\mathcal{C}$ a congruence of (G,Ω). The natural map $\Omega \to \Omega/\mathcal{C}$ induces a one-to-one correspondence between the congruences of (G,Ω) which contain $\mathcal{C}$ and the congruences of $(G,\Omega/\mathcal{C})$; it induces a one-to-one correspondence between the G-invariant subsets of Ω which are unions of $\mathcal{C}$ classes and the G-invariant subsets of $\Omega/\mathcal{C}$.*

THEOREM 1.4.3. *Let (G,Ω) be an ordered permutation group and N be a convex normal subgroup of G. Define $\mathcal{C}_N$ by: $\alpha \mathcal{C}_N \beta$ if $\alpha h_1 \le \beta \le \alpha h_2$ for some $h_1, h_2 \in N$. Then $\mathcal{C}_N$ is a congruence of (G,Ω) and $(G/N,\Omega/\mathcal{C}_N)$ is an action where $(\alpha\mathcal{C}_N)(gN) = (\alpha g)\mathcal{C}_N$ $(\alpha \in \Omega,\ g \in G)$. Conversely, if $\mathcal{C}$ is any congruence of (G,Ω), let $N = \{g \in G: \alpha g \mathcal{C} \alpha \text{ for all } \alpha \in \Omega\}$. Then*

N is a convex normal subgroup of G that is an ℓ-subgroup if (G,Ω) is an ℓ-permutation group. Moreover, in this case, $(G/N,\Omega/\mathcal{C})$ is an ℓ-permutation group where $Ng \geq Nf$ if $g \geq hf$ for some $h \in N$.

Proof: Suppose $\alpha\mathcal{C}_N\beta$ and let $g \in G$. Let $f \in N$ with $\alpha \leq \beta \leq \alpha f$ (without loss of generality). Then $\alpha g \leq \beta g \leq \alpha fg = \alpha g(g^{-1}fg)$. Since $N \lhd G$, $g^{-1}fg \in N$; so $\alpha g\mathcal{C}_N\beta g$. Hence $\mathcal{C}_N$ is a congruence of (G,Ω). If $Nh = Ng$, then $hN = gN$; so $g^{-1}h \in N$. But $\alpha h \leq \beta h \leq \alpha fh = (\alpha g)g^{-1}fg\cdot g^{-1}h$. Thus $(\beta\mathcal{C}_N)(Nh) = (\alpha\mathcal{C}_N)(Ng)$ if $\alpha\mathcal{C}_N\beta$ and $Nh = Ng$. It now follows easily that $(G/N,\Omega/\mathcal{C}_N)$ is an action (use Lemma 1.2.2).

Conversely, if $\mathcal{C}$ is a congruence of (G,Ω) and $N = \{g \in G: \alpha g\,\mathcal{C}\,\alpha \text{ for all } \alpha \in \Omega\}$, then clearly N is a subgroup of G. If $f \in G$ and $g \in N$, then as $(\alpha f^{-1})g\,\mathcal{C}(\alpha f^{-1})$ for all $\alpha \in \Omega$, $f^{-1}gf \in N$. Hence $N \lhd G$. Since N is the lazy subgroup of $(G,\Omega/\mathcal{C})$, $(G/N,\Omega/\mathcal{C})$ is faithful. The "ℓ-" part is now straightforward.

We next give some examples. The justifications of the claims in each are routine and are left to the reader.

EXAMPLE 1.4.4. Let G be a p.o. group and H a convex subgroup of G. Let $R(H)$ be the set of right cosets of H in G ordered by: $Hg_1 \leq Hg_2$ if $hg_1 \leq g_2$ for some $h \in H$. Then $R(H)$ is a partially ordered set. If $R(H)$ is a chain (under this ordering) then $(G,R(H),\theta)$ is an action where $(Hf)g\theta = Hfg$ $(f,g \in G)$. This action is faithful precisely when $\bigcap_{g \in G} g^{-1}Hg = \{e\}$. In the special case that $H = \{e\}$ and $R(H)$ is a chain (hence $G = R(H)$ is an o-group), the faithful transitive action (G,G,θ) is called the *right regular representation* of G. Examples 1.1.1 and 1.2.1 are right regular representations.

EXAMPLE 1.4.5. Let (G,Ω) be any ordered permutation group. Then (G,Ω) has two *trivial* congruences, denoted throughout the book by $\mathcal{S}$ and $\mathcal{U}$: $\alpha\mathcal{S}\beta$ if $\alpha = \beta$ (the *singleton* congruence)