Representations and characters of finite groups

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General representation theory

1 Basic concepts

Let G be an arbitrary finite group[†] and let K be an arbitrary field. Then a *(linear) representation* ρ *of* G *over* K is a homomorphism

$$\rho: G \rightarrow \mathrm{GL}(V)$$

where V is a finite dimensional vector space[†] over K and $\mathrm{GL}(V)$ is the group of nonsingular linear transformations of V into itself. If we are already given the vector space V, then we may refer to ρ as a representation of G *on* V. Although we shall normally write mappings on the right, we shall write $\rho(g)$ rather than $g\rho$ since we shall never consider compositions of representations; also, for $v \in V$, we shall write vg for $v \cdot \rho(g)$ if there is no risk of ambiguity.

If the dimension of V is n, we may choose a basis for V and identify V with the space K^n of n-tuples over K; then we may regard ρ as a map from G into $\mathrm{GL}(n, K)$, the group of nonsingular $n \times n$ matrices over K. The precise map so obtained depends on the choice of basis; thus a homomorphism

$$\rho: G \rightarrow \mathrm{GL}(n, K)$$

should be called a *matrix* representation. However, in a way which will be made precise shortly, matrix representations obtained by taking different bases are *similar*, and we shall move freely between representations on vector spaces and the corresponding matrix representations.

We shall study representations with two particular purposes in mind. The first is that a representation gives us something concrete, namely a group of linear transformations or matrices, to which the methods of linear algebra may be applied. The second is that by studying the values of the traces of the matrices $\rho(g)$, it may be possible to use the arithmetic properties of the field K to deduce information about an abstract group G. This is known as *character theory*, and much of this book will be devoted to this aspect in the case that K is the field of complex numbers $\mathbb{C}$.

[†] Throughout this book, groups will always be finite, with the obvious exception of groups of linear transformations, and vector spaces will be finite dimensional. However, most of the definitions of this section, although little of the subsequent theory, can be extended without these restrictions.

Such trace values are known as (ordinary) *characters*. However, in this chapter we shall develop the basic representation theory in the first spirit and in a form more general than that needed purely for character theory.

Examples (Groups and fields are arbitrary unless otherwise stated.)

1. Let V be a one-dimensional vector space over K. The map

$$g \rightarrow 1_V$$

for all $g \in G$ is the *trivial* representation of G over K.

2. Let G be a group which acts as a group of permutations on a finite set Ω, where $\Omega = \{e_1, \ldots, e_n\}$. Let V be a vector space of dimension n over K with a basis $\{v_1, \ldots, v_n\}$. For $g \in G$, let π_g be the linear transformation on V defined by the action on basis vectors

$$\pi_g : v_i \rightarrow v_j \quad \text{if and only if} \quad g : e_i \rightarrow e_j.$$

Then the map $\pi: G \rightarrow \mathrm{GL}(V)$ defined by $\pi(g) = \pi_g$ for all $g \in G$ is a *permutation* representation of G on V. Notice that the corresponding matrix representation (with respect to the basis $\{v_1, \ldots, v_n\}$) is given by permutation matrices.

3. Take $\Omega = G$ in Example 2 and define a permutation action by the mappings

$$g : x \rightarrow xg$$

for all $x, g \in G$. The associated representation is called the *right regular representation* of G.

4. Let N be a normal subgroup of G, and suppose that ρ is a representation of G/N. The mapping

$$\hat{\rho} : g \rightarrow \rho(gN)$$

for all $g \in G$ defines a representation on G. This representation is called the *inflation* of ρ.

Conversely, if σ is a representation of G such that N lies in the kernel of σ, then the mapping

$$\tilde{\sigma} : gN \rightarrow \sigma(g)$$

defines a representation of G/N.

5. If K is regarded as a one-dimensional vector space over itself, then multiplication acts as a linear transformation. Thus any homomorphism from a group G into the multiplicative group of K may be viewed as a representation. In particular, if ρ is a representation of G over K, then

the mapping

$$g \to \det(\rho(g))$$

for all $g \in G$ is a one-dimensional representation.

6. Let G be a cyclic group of order n and let g be a generator of G. Let ω be an nth root of unity in K (not necessarily primitive). Then the mapping

$$\rho_\omega: g^i \to \omega^i$$

defines a representation of G. Conversely, every one-dimensional representation of G is similar to a representation of this form.

7. The alternating groups A_4 and A_5 and the symmetric group S_4 are isomorphic, respectively, to the rotation groups of the regular tetrahedron, icosahedron and cube. By taking an orthonormal basis for $\mathbb{R}^3$, these isomorphisms lead to natural representations of the three groups by real *orthogonal* 3×3 matrices.

We shall now introduce some basic terminology. Let G and K be, as before, arbitrary and let ρ be a representation of G on a vector space V over K. The dimension $\dim_K(V)$ is called the *degree* of the representation ρ and will be denoted by $\deg \rho$. If the kernel, $\ker \rho$, of ρ is trivial, then ρ is *faithful*. If U is a subspace of V which is invariant under $\rho(g)$ for all $g \in G$, then U *admits* G, or is *G-invariant*. If $V \neq 0$ and the only G-invariant subspaces of V are 0 and V itself, then ρ is *irreducible*; otherwise ρ is *reducible*. If V can be written as the direct sum of two nonzero G-invariant subspaces, then ρ is *decomposable*; otherwise ρ is *indecomposable*.

It follows, trivially, that an irreducible representation is indecomposable. The converse is true provided that the characteristic of K does not divide† the order of G as we shall see in Section 3, but this need not be so in general. For example, a two-dimensional representation of the additive group of $\mathbb{Z}_p$ over $\mathbb{Z}_p$ is given by

$$t \to \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

and this is indecomposable but not irreducible: the subspace spanned by the second basis vector is invariant, but not complemented.

Suppose that ρ_1 and ρ_2 are two representations of G over K on vector spaces V_1 and V_2 respectively. Then ρ_1 and ρ_2 are said to be *equivalent* if there exists an isomorphism $\sigma: V_1 \to V_2$ such that

$$\rho_2(g) = \sigma^{-1}\rho_1(g)\sigma \quad \text{for all } g \in G;$$

we shall write $\rho_1 \sim \rho_2$, or $\rho_1 \sim_K \rho_2$ if we wish to emphasise the field K.

†This will always be understood to include the case that char $K = 0$.

Visibly, this defines an equivalence relation on the representations of a group (over a fixed field), and by a set of *distinct* representations of a group, we shall always mean a collection of inequivalent representations.

If $V_1 = V_2$, then this definition can be applied to two different representations of G on V_1; also, if ρ is a representation of G on V and ρ_1 and ρ_2 are the associated *matrix* representations with respect to different bases of V, then immediately ρ_1 and ρ_2 are equivalent. In this case, there exists a nonsingular matrix X such that

$$\rho_2(g) = X^{-1}\rho_1(g)X$$

for all $g \in G$, and we say that ρ_1 and ρ_2 are *similar*.

Exercises

1. Show that the derived group G' of a group G lies in the kernel of any representation of G of degree 1. Deduce that, if $\rho: G \rightarrow \mathrm{GL}(n, K)$ is a matrix representation of G, then $\rho(g) \in \mathrm{SL}(n, K)$ whenever $g \in G'$.
2. Let ρ_1 and ρ_2 be equivalent representations of a group G. Show that, whenever $g \in G$, the linear transformations $\rho_1(g)$ and $\rho_2(g)$ have the same minimal and characteristic polynomials. If g is an element of order n, show that the minimal polynomial of $\rho_1(g)$ divides $x^n - 1$.
3. Let ρ be a representation of a group G over an algebraically closed field K whose characteristic does not divide $|G|$. If g is a fixed element of G, show that there exists a basis with respect to which $\rho(g)$ has a diagonal matrix.
4. Let G be a finite abelian group and let K be an algebraically closed field of characteristic not dividing $|G|$. If G has a representation on a vector space V over K, show that there exists a basis for V with respect to which every element of G is represented by a diagonal matrix. Deduce that every irreducible representation of G over K has degree 1.
5. Let G and K be as in Exercise 4 and regard the irreducible representations as maps from G to K. Suppose that G has a decomposition as the direct product of cyclic subgroups generated by elements $g_1, \ldots, g_r$. Show that an irreducible representation ρ of G is determined by its values on the elements $g_1, \ldots, g_r$ alone. Deduce that the number of distinct irreducible representations of G over K is $|G|$.

   Show that the set of distinct irreducible representations forms an abelian group G^* under composition defined by

   $$(\rho_1 \cdot \rho_2)(g) = \rho_1(g) \cdot \rho_2(g) \quad \text{for all } g \in G,$$

   and that G^* is isomorphic (as an abstract group) to G.

6. Show that an irreducible representation of a cyclic group G of prime order p over a field of characteristic p is trivial. By considering the possible Jordan canonical forms for a linear transformation of order p, determine a complete set of inequivalent indecomposable representations of G over a field of characteristic p.
7. Determine the irreducible representations of an arbitrary p-group over a finite field K of characteristic p.
[Hint. Show that the subgroup of upper triangular matrices in $\mathrm{GL}(n, K)$ is a Sylow subgroup, where n is the degree, and apply Sylow's theorem.]
8. Let G be the dihedral group D_{2n} of order $2n$, the group of symmetries of a regular n-gon. Then G has a presentation
$$G = \langle x, y \mid x^n = y^2 = 1, y^{-1}xy = x^{-1} \rangle.$$
Suppose that n is odd. Show that $G' = \langle x \rangle$, and hence determine the (two) one-dimensional representations of G over $\mathbb{C}$.
Use Exercises 1 and 4 to show that, if $\rho: G \to \mathrm{GL}(2, \mathbb{C})$ is a two-dimensional irreducible complex representation of G, then $\rho \sim \rho_1$ where
$$\rho_1(x) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}$$
and ω is a nonidentity nth root of unity. Determine which matrices of order 2 can invert $\rho_1(x)$, and hence show that $\rho_1 \sim \rho_2$ where
$$\rho_1(x) = \rho_2(x) \quad \text{and} \quad \rho_2(y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$
Deduce that the number of inequivalent irreducible complex representations of G of degree 2 is $\frac{1}{2}(n-1)$.
[Notice that $\frac{1}{2}(n-1)\cdot 2^2 + 2\cdot 1^2 = 2n$: see Exercise 17 of Section 2 and also Corollary 20 (iii).]
9. Carry out the corresponding analysis to Exercise 8 when n is even. [Note that, in this case, $G' = \langle x^2 \rangle$.]
10. Let G be the generalised quaternion group of order 2^{n+1} ($n \geqslant 2$) which has a presentation
$$g = \langle x, y \mid x^{2^n} = 1, y^2 = x^{2^{n-1}}, y^{-1}xy = x^{-1} \rangle.$$
Show that there is a complex representation $\rho: G \to \mathrm{GL}(2, \mathbb{C})$ for which
$$\rho(x) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \quad \text{and} \quad \rho(y) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
where ω is a primitive 2^nth root of unity and that ρ is faithful and

irreducible. Show also that every element of G is represented by a matrix in $SL(2,\mathbb{C})$, and deduce directly that G contains a unique involution (element of order 2).

11. In Example 7, use geometrical considerations to show that the representations defined there are irreducible.
12. Let G be a group, and define an action of G as a group of permutations on itself by $g: x \to g^{-1}x$. Show that this gives rise to a representation over any field, called the *left regular representation*.
13. Show that, if ρ and σ are similar matrix representations of a group G over a field K, then $\mathrm{tr}(\rho(g)) = \mathrm{tr}(\sigma(g))$ for all $g \in G$.

    Show also that $\mathrm{tr}(\rho(g)) = \mathrm{tr}(\rho(h))$ whenever g and h are conjugate elements in G. (This says that we have defined a *class function* on G.) [tr denotes the *trace* of matrix: if $A = (a_{ij})$, then $\mathrm{tr}\, A = \sum_i a_{ii}$.]
14. For each of the groups A_4, A_5 and S_4, determine the value of $\mathrm{tr}(\rho(g))$ for a representative of each conjugacy class, where ρ is the three-dimensional real representation defined for each of the three groups in Example 7.

2 Group rings, algebras and modules

Let G be a finite group and let ρ be a representation of G on a vector space V over a field K. Then the K-linear combinations of the linear transformations $\rho(g)$ for $g \in G$ form a subring of the full ring $\mathscr{L}(V)$ of linear transformations of V. The vector space V can be given the structure of a right module over this subring. We shall formalise this, but make our first definition more general.

Let G be a group and let R be a commutative ring with identity. Then the *group ring* RG consists of the set of all formal sums

$$\sum_{g \in G} a_g g \quad (a_g \in R)$$

together with the binary operations

$$\sum_{g \in G} a_g g + \sum_{g \in G} b_g g = \sum_{g \in G} (a_g + b_g) g \quad (a_g, b_g \in R)$$

$$\left(\sum_{g \in G} a_g g\right) \cdot \left(\sum_{h \in G} b_h h\right) = \sum_{g \in G} \left(\sum_{h \in G} a_{gh^{-1}} b_h\right) g$$

$$= \sum_{g,h \in G} (a_g b_h)(gh)$$

where gh is the group product in G. It is a straightforward calculation to verfiy that RG is an associative ring with identity. If R is a field K, then KG has the structure of a vector space over K as well as that of ring. So in this case KG is a K-algebra of finite dimension $|G|$, called the *group algebra* of G over K.

Let $\rho: G \to \mathrm{GL}(V)$ be a representation of G on a vector space V over the field K. Then we can extend ρ by linearity to a K-algebra homomorphism

$$\rho: KG \to \mathrm{Hom}_K(V, V).$$

The extension ρ is a representation of the algebra KG. This gives V the structure of a unitary right KG-module under the operation

$$v(\sum a_g g) = \sum a_g(v \cdot \rho(g)),$$

called the *representation module*. Since $\dim_K(V)$ is finite, V certainly satisfies both chain conditions as a KG-module; the composition factors (or the representations that they *afford*–see Propositions 1 and 2 below) are called the *irreducible constituents* of ρ. Conversely, given a unitary right KG-module M which is finite dimensional as a vector space over K, we may obtain a representation σ of G defined by $m \cdot \sigma(g) = mg$. In our approach, we shall switch freely between modules and representations.

These considerations extend naturally to the representations (over K) of an arbitrary K-algebra A[†]. Conversely, given an A-module M[†], we may recover a representation of A. It is easy to see that the definitions of Section 1 extend to representations of K-algebras (over K) and then to verify the following.

Proposition 1. *Let G be a group and K a field. Then a representation of G over K is irreducible or indecomposable if and only if the same is true of the corresponding representation of KG. Two representations of G are equivalent if and only if the same is true for the corresponding representations of KG.*

Proposition 2. *Let A be an algebra over a field K and let M be an A-module which affords a representation ρ of A. Then*

(i) *M is irreducible as an A-module if and only if ρ is an irreducible representation,*

(ii) *M is indecomposable as an A-module if and only if ρ is an indecomposable representation, and*

(iii) *if $M = M_1 \oplus \cdots \oplus M_n$ is a direct sum decomposition of M into a sum of indecomposable submodules, then n and the isomorphism classes of $M_1, \ldots, M_n$ are uniquely determined.*

[†] Algebras will always be associative with identity and finite dimensional. Modules will be unitary right modules unless explicitly stated otherwise, and modules over algebras will be finite dimensional over the underlying field.

Furthermore, if N is an A-module affording a representation σ, then ρ and σ are equivalent if and only if M and N are isomorphic as A-modules.

Proof. All but (iii) are immediate from the definitions. We note that M always has such a decomposition since it satisfies both chain conditions on submodules: then the Krull–Schmidt theorem holds.

We shall now consider some constructions of representations that will be used later, leaving the necessary verifications as exercises.

Examples

1. If $\rho: G \to \mathrm{GL}(V)$ is a representation of G and H is a subgroup of G, then the *restriction* $\rho|_H: H \to \mathrm{GL}(V)$ is a representation of H. Since $KH \subseteq KG$, a KG-module M has the structure of a KH-module, which we denote by M_H.

2. If M_1 and M_2 are KG-modules affording representations ρ_1 and ρ_2 respectively, then their direct sum $M_1 \oplus M_2$ affords the *sum* $\rho_1 + \rho_2$.

3. Let M be a KG-module affording a representation ρ, and let M^* be the dual of M as a vector space over K. Then M^* may be given the structure of a KG-module as follows. For $m^* \in M^*$ and $g \in G$, define

$$m^*g = \rho(g^{-1}) \circ m^*;$$

then $m^*g \in M^*$ and we put

$$m^*(\sum a_g g) = \sum a_g(m^*g).$$

M^* affords the *contragredient representation* ρ^*: as matrix representations with respect to dual bases, $\rho^*(g)$ will be the transpose inverse of $\rho(g)$ for each $g \in G$.

4. Let $\rho: G \to \mathrm{GL}(n, K)$ be a matrix representation, and suppose that α is an automorphism of K. If $\rho(g) = (a_{ij}(g))$, then a representation ρ^α can be defined by putting $\rho^\alpha(g) = ((a_{ij}(g))^\alpha)$. An example of particular importance occurs when $K = \mathbb{C}$ and α is complex conjugation.

5. Let ρ be a representation of a group G and let θ be an automorphism of G. Then the map $g \to \rho(g\theta)$ defines a representation of G. (This is just the composite of θ and ρ.) Notice that this action of θ on sets of representations preserves equivalence.

6. Let G and H be groups and let M and N be KG- and KH-modules respectively affording representations ρ and σ of G and H. Let $M \otimes N$ be the tensor product of M and N as vector spaces. Then $M \otimes N$ can be

given the structure of a $K(G \times H)$-module by defining

$$(m \otimes n)(g, h) = mg \otimes nh$$

for decomposable tensors and extending by linearity. $M \otimes N$ affords the *tensor product* representation $\rho \otimes \sigma$ of $G \times H$.

7. Take $G = H$ in Example 6 above and restrict $\rho \otimes \sigma$ to the diagonal subgroup of $G \times G$. Then $M \otimes N$ affords a representation $\rho \otimes \sigma$ of G, also called the tensor product, under the operation

$$(m \otimes n)g = mg \otimes ng.$$

8. Let M be an A-module, where A is a K-algebra, and suppose that L is an extension of K. Put $A^L = L \otimes_K A$ and $M^L = L \otimes_K M$. Then A^L may be regarded as an L-algebra and M^L as an A^L-module under the operation

$$(l \otimes m)(l' \otimes a) = ll' \otimes ma.$$

Notice that $\dim_L(M^L) = \dim_K(M)$. In the case of group algebras, there is a natural isomorphism between $(KG)^L$ and LG. Regarding M^L as an LG-module and letting M and M^L afford representations ρ and ρ^L of G respectively, the corresponding matrix representations of G will be identical with respect to bases of the form $\{m_i\}$ and $\{1 \otimes m_i\}$. Thus, by abuse of terminology, $\rho \sim \rho^L$ and characteristic and minimal polynomials are unaltered by field extension. Usually we shall follow this abuse and refer to *extending the ground field*, rather than actually perform the tensor product construction since this is what happens in reality when considering representations of groups and embedding $\mathrm{GL}(n, K)$ in $\mathrm{GL}(n, L)$.

If ρ is an irreducible representation of G over K, then ρ is *absolutely irreducible* if ρ^L is irreducible whenever L is an extension of K, and K is a *splitting field* for G if every irreducible representation of G over K is absolutely irreducible. As will be seen in Exercise 12, any extension of a splitting field is a splitting field: no 'new' irreducible representations occur in a larger field. These definitions extend to algebras and modules provided the precise formulation is taken.

We shall return to the study of absolute irreducibility in Section 4.

Exercises

1. If G is a group and N is a normal elementary abelian p-subgroup of G, show that N may be given the structure of a $\mathbb{Z}_p(G/N)$-module by defining

$$(n)gN = g^{-1}ng$$

for all $n \in N$ and $g \in G$. What is the kernel of the corresponding representation of G?

2. Show that, if G is a group and M and N are normal subgroups of G with $N \subset M$ and M/N an elementary abelian p-group, then M/N may be given the structure of a $\mathbb{Z}_pG$-module.
3. Show that if H is a subgroup of a group G and M and M' are KG-modules, then $(M \oplus M')_H \cong M_H \oplus M'_H$ and $(M \otimes M')_H \cong M_H \otimes M'_H$. Show also that $(M^*)_H \cong (M_H)^*$.
4. Show that the module given by a permutation representation of a group always contains a submodule affording the trivial representation.
5. Show that a group ring KG, viewed as a module over itself, has a unique submodule affording the trivial representation, spanned (as a vector space) by $\sum g$.
6. Show that a group ring KG affords the right regular representation of G when viewed as a right module over itself. What can be said of KG as a *left* module?
7. Let H be a normal subgroup of a group G and let K be a field. Show that KH can be given the structure of a KG-module by defining

$$h \cdot g = g^{-1}hg$$

and extending linearly.
8. Let A be a K-algebra. Show that, as a module over itself, the submodules of A are precisely the right ideals of A.
9. Let A be a K-algebra and let M be an irreducible A-module. Show that A has a maximal right ideal B such that $A/B \cong M$ as an A-module. Deduce that A has only finitely many inequivalent irreducible representations.
10. Let A be a K-algebra. Show that the intersection of all maximal right ideals forms a two-sided ideal $J(A)$, called the *radical* of A, and that

$$J(A) = \{a \in A \mid Ma = 0 \text{ for every irreducible } A\text{-module } M\}.$$

Show also that $J(A/J(A)) = 0$.
11. Let M be an irreducible KG-module where char K divides $|G|$. For $m \in M$, show that, if $m(\sum g) \neq 0$, then $m(\sum g)$ spans a subspace on which G acts trivially. Deduce that, in fact, $m(\sum g) = 0$ and hence that $J(KG) \neq 0$.
12. Let A be a K-algebra and let M be an A-module. Suppose that L is an extension of K. Show that

   (i) if N is an A-submodule of M, then N^L may be regarded as an A^L-submodule of M^L and $M^L/N^L \cong (M/N)^L$,
   (ii) if E is an extension of L, then $M^E \cong (M^L)^E$, and
   (iii) if M is absolutely irreducible, then so is M^L.

Suppose that K is a splitting field for A and that $\{M_1, \ldots, M_r\}$ is a complete set of nonisomorphic irreducible A-modules. If $\tilde{M}$ is an irreducible A^L-module, show that $\tilde{M} \cong M_i^L$ for some i and deduce that L is also a splitting field for A.
[Note. This is also a consequence of theorems that will be proved later, but it is instructive to find an elementary proof. It will also follow from the structure theorems that no two of $M_1^L, \ldots, M_r^L$ are isomorphic. See Theorem 13 and Exercise 8 of Section 4.]

13. Let ρ be a representation of a group G over a field K, where char K does not divide $|G|$. Show that there exists a finite extension L of K such that ρ is similar over L to a representation in which any specified element g may be represented by a diagonal matrix. What restriction need be imposed to obtain a like conclusion if char K does divide $|G|$?
14. Let G be an abelian group of exponent n. Show that if K is a field whose characteristic does not divide $|G|$ and K contains primitive nth roots of unity, then K is a splitting field for G. Show that if the second condition is not satisfied, then K is not a splitting field.
[See Exercises 4 and 5 of Section 1. The *exponent* of a group is the least common multiple of the orders of its elements.]
15. Let G a group and H a subgroup, and let K be a field. Show that, if $\{g_1, \ldots, g_n\}$ is a set of coset representatives for H in G, then the subspace of KG spanned by the coset sums
$$\sum_{h \in H} hg_i, \quad i = 1, \ldots, n,$$
forms a submodule.
16. Determine the irreducible constituents of the right regular representation of the symmetric group S_3 over each of the fields $\mathbb{C}, \mathbb{Q}, \mathbb{Z}_2$, and $\mathbb{Z}_3$ and deduce that each is a splitting field for S_3.
17. Let $G = D_{2n}$ and suppose that n is odd. Using the notation of Exercise 8 of Section 1, show that, if ω is a nonidentity nth root of unity, the complex group algebra $\mathbb{C}G$ has a two-dimensional subspace on which x acts as multiplication by ω in the right regular representation. By considering the action of y on the four-dimensional subspace spanned by the eigenspaces of x corresponding to eigenvalues ω and ω^{-1}, show that $\mathbb{C}G$ has at least two composition factors affording the representation ρ_2. Deduce, by considering the sum of their dimensions, that $\mathbb{C}G$ has exactly one composition factor affording each one-dimensional representation and exactly two composition factors affording each two-dimensional irreducible representation.

[Note. Compare this with the previous exercise. Exercise 17 now shows that every irreducible representation of G over $\mathbb{C}$ has been described. See also Corollary 20(ii).]

18. Obtain the analogue of the previous exercise with n even.
19. Show that A_5 has two inequivalent real representations of degree 3. [Hint. Consider the effect of conjugation by a 4-cycle in S_5 on a 5-cycle that it normalises, and trace values. See Example 7 and Exercise 14 of Section 1.]
20. Let M and N be A-modules for a K-algebra A. Show that $\mathrm{Hom}_K(M, N)$ can be given the structure of an A-module by defining
$$m(fa) = (mf)a$$
for $m \in M, f \in \mathrm{Hom}_K(M, N)$ and $a \in A$, where $\mathrm{Hom}_K(M, N)$ denotes the vector space of linear transformations from M to N.
21. Let M and N be KG-modules. Show that $\mathrm{Hom}_K(M, N)$ can be given the structure of a KG-module by defining a map f^g by
$$mf^g = (mg^{-1}f)g$$
for all $m \in M$ and extending by linearity. Show that the space of fixed points of the action of G on $\mathrm{Hom}_K(M, N)$ is $\mathrm{Hom}_{KG}(M, N)$, the set of homomorphisms from M to N as KG-modules.
22. Let K be an arbitrary field. Show that any $n \times n$ matrix over K can be expressed as a finite K-linear combination of nonsingular matrices over K.
[In this sense, there exists a homomorphism from the group ring of $\mathrm{GL}(n, K)$ over K, taking only finitely many nonzero terms, onto the full matrix ring $\mathcal{M}_n(K)$.]
23. Let M be an A-module and define the *symmetric* and *antisymmetric* subspaces of $M \otimes M$ as the subspaces M_S and M_A spanned by the sets $\{m \otimes m \mid m \in M\}$ and $\{(m_1 \otimes m_2 - m_2 \otimes m_1) \mid m_1, m_2 \in M\}$ respectively. Show that M_S and M_A are submodules and that $M \otimes M = M_S \oplus M_A$ if $\mathrm{char}\, K \neq 2$.

3 Complete reducibility

Let K be an arbitrary field and let A be a K-algebra. An A-module M is said to be *completely reducible* if the conditions of the following proposition hold. The representation of A afforded by M will also be said to be completely reducible.

Proposition 3. *Let M be an A-module. Then the following three conditions are equivalent.*

(i) *M is a direct sum of irreducible submodules.*
(ii) *M is a sum of irreducible submodules.*
(iii) *Every submodule of M is a direct summand.*

Proof. Clearly (i) implies (ii). If (ii) holds, we establish (iii) by induction on codimension. Let N be a submodule of M. If $N = M$, there is nothing to prove; otherwise, by (ii), there exists an irreducible submodule L of M such that $L \cap N = 0$. Then $N + L = N \oplus L$ and, by induction, there exists a submodule L' of M such that

$$M = (N \oplus L) \oplus L' = N \oplus (L \oplus L').$$

If (iii) holds, let $\{M_1, \ldots, M_n\}$ be a collection of irreducible submodules of M which generate their direct sum M'. If $M' \neq M$, then we can choose a submodule M'' of M and an irreducible submodule M_{n+1} of M'' such that $M = M' \oplus M''$ and

$$(M_1 \oplus \cdots \oplus M_n) + M_{n+1} = M_1 \oplus \cdots \oplus M_n \oplus M_{n+1} \supset M_1 \oplus \cdots \oplus M_n.$$

Since M is finite dimensional over K, it follows that M is a direct sum of irreducible submodules.

For the remainder of this section, we shall restrict our attention to representations of group algebras. The next result will provide a fundamental dichotomy according to the characteristic of the underlying field.

Theorem 4 (Maschke). *Let G be a finite group and let K be a field whose characteristic does not divide $|G|$. Then every KG-module is completely reducible.*

Proof. Suppose that M is a reducible KG-module and let U be a proper, nonzero submodule. Then we must show that U is a direct summand of M. Let V be a complement to U in M, viewed only as a vector space, and let $\theta: M \to V$ be the corresponding projection. We apply an averaging process to find a KG-invariant complement to U.

Define a map $\varphi: M \to M$ by the formula

$$m\varphi = |G|^{-1} \sum_{g \in G} ((mg)\theta)g^{-1};$$

clearly φ is K-linear. Put $W = M\varphi$. Then $M = U + W$ since, if $m \in M$,

$$m - m\varphi = |G|^{-1} \sum_{g \in G} (mg - (mg)\theta)g^{-1} \in U.$$

Also, if $h \in G$,

$$\begin{aligned}(m\varphi)h &= |G|^{-1} \sum_{g \in G} ((mg)\theta)g^{-1}h \\ &= |G|^{-1} \sum_{g \in G} ((mh(g^{-1}h)^{-1})\theta)g^{-1}h \\ &= |G|^{-1} \sum_{x \in G} ((mhx)\theta)x^{-1} \\ &= (mh)\varphi\end{aligned}$$

where we put $x = (g^{-1}h)^{-1}$. So W is G-invariant. Now $\varphi|_U = 0$ since $\theta|_U = 0$. Hence, as φ is K-linear,

$$\dim_K(U) + \dim_K(W) \leqslant \dim_K(M),$$

and so $M = U \oplus W$ as a KG-module.

If the characteristic of K does divide $|G|$, then KG is not completely reducible; we shall leave a proof to the exercises below. After the general considerations of this chapter, we shall be restricting our attention to the *ordinary* representation theory of finite groups–namely, that over the complex field, or at least a splitting field of characteristic zero. Here, the complete reducibility of the group algebra will be crucial in developing the basic formulae on which character theory will depend. Analogous results for other fields whose characteristic does not divide the group order hold, but are of limited interest. If the characteristic of the field does divide the group order, then one speaks of *modular* representation theory as was first fully explored by Brauer after earlier work by Dickson. In particular, Brauer developed an extensive theory on the connection between ordinary and modular representations. (Strictly speaking, modular representation theory refers to the situation for fields of nonzero characteristic but, as remarked above, in the coprime characteristic case, this is the same as the ordinary theory.)

Exercises

1. Show that every submodule and factor module of a completely reducible module is completely reducible. Show also that the intersection of all maximal submodules of a completely reducible module is the zero submodule. Deduce that if K is a field whose characteristic divides the order of a group G, then KG is not completely reducible as a KG-module. (See Exercise 11 of Section 2).
   [Note. This shows that the conclusion of Maschke's theorem would be false if the hypothesis about the field characteristic were omitted. See also Exercise 6.]

2. Show that any KG-module has a unique submodule maximal with respect to admitting G trivially.
3. Let A be a K-algebra and let M be an A-module. Define the *socle* $\mathrm{soc}(M)$ to be the sum of all the irreducible submodules of M and the *radical* $\mathrm{rad}(M)$ to be the intersection of all maximal submodules of M. Prove that $\mathrm{soc}(M)$ and $M/\mathrm{rad}(M)$ are completely reducible.
4. Let A be a K-algebra and let B be a right ideal of A. Prove that A/B is completely reducible as an A-module if and only if $B \supseteq J(A)$. [See Exercise 10 of Section 2.]
5. Show that the set of elements $\sum a_g g$ in a group algebra KG for which $\sum a_g = 0$ forms an ideal $A(KG)$, called the *augmentation ideal.* Show also that $KG/A(KG)$ is isomorphic to the trivial KG-module.
6. Show that, if $\mathrm{char}\, K$ divides $|G|$, then $A(KG)$ contains the unique trivial submodule of KG. Deduce that, in this case, KG has at least two trivial constituents as a KG-module. [Note. KG contains a unique trivial submodule by Exercise 5 of Section 2.]
7. Verify directly that the group algebra of the symmetric group S_3 is completely reducible for the fields $\mathbb{C}$ and $\mathbb{Q}$, but not $\mathbb{Z}_2$ or $\mathbb{Z}_3$. [Use the computations from Exercise 16 of Section 2.]
8. Let A be a K-algebra and let M be an A-module. Identify $\mathrm{Hom}_K(M, M)$ with the full matrix algebra $\mathcal{M}_n(K)$ where $n = \dim_K(M)$ and let A act on $\mathcal{M}_n(K)$ via the formula $m(fa) = (mf)a$. By considering the submodules consisting of matrices with nonzero entries only in a single row, show that $\mathrm{Hom}_K(M, M)$ is isomorphic as an A-module to a direct sum of n copies of M. Deduce that, if M is irreducible, then $\mathrm{Hom}_K(M, M)$ is completely reducible.
9. Let R be an arbitrary ring. Show that the equivalence of the three statements in Proposition 3 holds for R-modules, with no requirement of a finiteness condition (in that case, finite dimensionality).
10. Let P be an elementary abelian p-group and let H be a p'-group which acts on P. By viewing P as a $\mathbb{Z}_p H$-module, prove that $P = C_P(H) \times [P, H]$. [A *p'-group* is a group of order not divisible by a prime p.]

4 Absolute irreducibility and the realisation of representations

We begin by studying an irreducible module M for a K-algebra A and its endomorphism ring $\mathrm{Hom}_A(M, M)$. Our initial goal will be to determine a criterion for M to be absolutely irreducible, namely that only the scalar transformations commute with the action of A. Then we will specialise to the study of group rings and their splitting fields in characteristic 0. The approach that we take is via the double centraliser lemma (Theorem 7); this will in fact yield the full structure of the irreducible representations as a consequence, and we shall obtain this in the next section.

In the following result, we shall identify the field K with the space of scalar transformations: we shall often do this without further comment. The first part is known as *Schur's lemma.*

Theorem 5. $\mathrm{Hom}_A(M, M)$ *is a division ring. If K is algebraically closed, then* $\mathrm{Hom}_A(M, M) = K$.

Proof. Since M is irreducible, every endomorphism is either zero or bijective and, in the latter case, it is a trivial verification to show that inverses are also endomorphisms.

Let $\varphi \in \mathrm{Hom}_A(M, M) - \{0\}$. Then φ has a minimal polynomial $f(x)$ since M is finite dimensional over K. If K is algebraically closed, we may write

$$f(x) = \prod_{i=1}^{m} (x - a_i)$$

for some $a_1, \ldots, a_m \in K$. As $(\varphi - a_i 1)$ is either zero or invertible for each i, it follows that $\varphi = a_i 1$ for some i.

We now investigate the structure of M further. Let A_M denote the image of A in $\mathrm{Hom}_K(M, M)$. Then M is irreducible as an A_M-module. Since $\mathrm{Hom}_K(M, M)$ is completely reducible as an A-module, so is A_M, and A_M is isomorphic to a direct sum of copies of M. (See Exercise 8 of Section 3.) Now suppose that I is a proper two-sided ideal of A_M. Then A_M contains a minimal right ideal M_0, isomorphic to M as an A_M-module, with $I \cap M_0 = 0$. So $M_0 I \subseteq M_0 \cap I = 0$, and hence $I = 0$ since A_M acts faithfully on M. Thus we have shown the following.

Lemma 6. A_M *is a simple K-algebra.*

Now let $D = \mathrm{Hom}_A(M, M)$. Then M has the structure of a right D-module. Identifying K with the scalar transformations, we see that K lies in the centre of D and hence that $\mathrm{Hom}_D(M, M) \subseteq \mathrm{Hom}_K(M, M)$.

Theorem 7 (Double centraliser lemma). $\mathrm{Hom}_D(M, M) = A_M$.

Proof. Without loss, we may suppose that $A = A_M$ and that $M \subseteq A$. From the definition of D we know that $A \subseteq \mathrm{Hom}_D(M, M)$, so we need only establish the reverse inclusion.

Let $\theta \in \mathrm{Hom}_D(M, M)$. For each $m \in M$, define a map $\theta_m: M \to A$ by $x\theta_m = mx$. As M is a right ideal of A, in fact $\theta_m: M \to M$; then $\theta_m \in D$ since, whenever $a \in A$ and $x \in M$,

$$(xa)\theta_m = m(xa) = (mx)a = (x\theta_m)a.$$

Thus, if $m, n \in M$,

$$(mn)\theta = (n\theta_m)\theta = (n\theta)\theta_m = m(n\theta). \tag{4.1}$$

Let $n \in M - \{0\}$. Since A is simple and has an identity, $AnA = A$ so that we can write

$$1 = \sum_i a_i n b_i \tag{4.2}$$

for suitable elements $\{a_i, b_i\}$ in A. So, whenever $m \in M$, we obtain, using (4.1) and (4.2), the formula

$$m\theta = \sum((ma_i)(nb_i))\theta = \sum(ma_i)((nb_i)\theta) = m\sum a_i((nb_i)\theta).$$

Thus θ acts via right multiplication by an element of A; that is, $\theta \in A$.

If we identify $\mathrm{Hom}_K(M, M)$ with the full matrix algebra $\mathscr{M}_n(K)$ where $n = \dim_K(M)$ and let L be an extension of K, we can see that $\mathrm{Hom}_L(M^L, M^L)$ may be identified with $L \otimes_K \mathrm{Hom}_K(M, M)$. So we obtain the following.

Corollary 8. *Suppose that* $\mathrm{Hom}_A(M, M) = K$. *Then* $A_M = \mathrm{Hom}_K(M, M)$. *If* L *is an extension field of* K, *then* $(A_M)^L = \mathrm{Hom}_L(M^L, M^L)$ *and* $\mathrm{Hom}_{A^L}(M^L, M^L) = L$.

We are now in a position to establish a criterion for absolute irreducibility. This involves only the module M, and does not require a consideration of extension fields.

Theorem 9. *M affords an absolutely irreducible representation of A if and only if* $\mathrm{Hom}_A(M, M) = K$.

Proof. Suppose that $\mathrm{Hom}_A(M, M) = K$. If L is an extension field of K, then in a natural sense $\mathrm{Hom}_L(M^L, M^L) \supseteq (A^L)_{M^L} \supseteq (A_M)^L$ so that, by Corollary 8,

$$(A^L)_{M^L} = \mathrm{Hom}_L(M^L, M^L).$$

So certainly M^L is irreducible as an A^L-module.

Conversely, suppose that $\mathrm{Hom}_A(M, M) \neq K$. Let $\varphi \in \mathrm{Hom}_A(M, M) - K$, and let L be an extension field of K which contains a root λ of the