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Groups acting on graphs

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To Alan, Elena and Xaro
To Jean, Luke and Megan

Contents

Preface

The simplest instance of the interplay between group theory and topology occurs where a group acts on a graph and information is obtained about the group or about the graph; it is these occurrences which form the theme of this book.

Chapter I offers a review of the Bass–Serre theory of groups acting on trees and graphs, with some typical combinatorial group theoretic applications. For the sake of novelty, we have included a very recent result from the literature on the fixed group of an automorphism of a free group. Although we have attempted to make the account self-contained, it is rather brusque for initiation purposes, and the reader should ideally already have some familiarity with group theory, group actions, presentations and combinatorial theory.

Chapters II, III and IV are essentially new.

Chapter II, using Boolean rings, associates to each connected graph and positive integer n, a tree which explains how the graph disconnects when any n edges are deleted. One application is the recent result from the literature characterizing infinite finite-valency distance-transitive graphs. This chapter is elementary in the sense that no background material is assumed.

Chapter III is devoted to drawing lines joining up functions in an equivariant way to get a previously unsuspected tree. The argument is technical and elementary. The result has some rather pleasing applications, which are collected together in Chapter IV. New results include the proof of a conjecture of Wall, and a characterization of arbitrary groups with more than one end; previously known results are the characterization of groups of cohomological dimension at most one over an arbitrary ring, and the characterization of groups which have a free subgroup of finite index. The reader is assumed to be familiar with module theory and exact sequences; a Sylow theorem is used in a remark; cohomology is introduced in a mild way, since the results can be phrased in terms of derivations to projective modules.

Chapters V and VI examine dimensions two and three, and consist of results which, having only recently appeared in the literature, are appearing in book form for the first time.

Chapter V is an algebraic account of the cohomological characterization of infinite surface groups as the groups which satisfy two-dimensional Poincaré duality. The cohomology and topology are more sophisticated than in the rest of the book. The reader is assumed to be familiar with the necessary homological algebra, which is quickly summarized without many proofs. The reader familiar with the topology of surfaces, or manifolds in general, will be able to appreciate the motivation behind the entire chapter; the reader without the background in topology will have to be sufficiently algebraically inclined to be motivated by the result in its own right.

Chapter VI examines groups acting on two-complexes and deduces that almost finitely presented groups are accessible in the sense of Wall. It concludes with a similar analysis of three-manifolds and deduces the equivariant loop and sphere theorems. Here the topological background is summarized without proofs.

There are no exercises, apart from four open conjectures and the occasional tedious argument left to the reader.

Each chapter concludes with some notes and comments citing our sources for results and ideas. The sources, which are listed in the absolutely minimal bibliography and author index, tend not to be primary, and our attributions should be taken lightly, especially by authors who have been omitted.

We are indebted to the many mathematicians who have made helpful comments and devoted much time and effort to helping us understand the literature; their sole reward is the knowledge that the book would have been even worse without their help.

We thank Ed Formanek and Peter Linnell for generously contributing unpublished results and arguments.

The first-named author thanks the Mathematics Department of Pennsylvania State University for providing a graduate course forum to air and develop some of the results, and the CRM in Barcelona for support and gracious hospitality during the summer of 1985. The second-named author thanks the Royal Society and the IHES for support and gracious hospitality during the first six months of 1983.

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Conventions

G denotes a group, fixed throughout the book.

$\varnothing$ denotes the empty set.

Sets are indicated by $\{x|x\cdots\}$ or sometimes $\{x:x\cdots\}$ for typographical reasons.

$B \subseteq A$ means B is a subset of A.

$B \subset A$ means B is a *proper* subset of A, that is, distinct from A.

If $B \subseteq A$ then $A - B$ denotes the complement of B in A.

$A \cup B$, $A \vee B$, $A \cap B$, $A \times B$, respectively, denote the union, the disjoint union, the intersection and the Cartesian product of two sets, A, B.

$\bigcup_{i \in I} A_i$, $\bigvee_{i \in I} A_i$, $\bigcap_{i \in I} A_i$, $\prod_{i \in I} A_i$, respectively, denote the union, the disjoint union, the intersection and the Cartesian product of a family of sets A_i indexed by the elements i of a set I.

A^n denotes the Cartesian product of copies of a set A indexed by a non-negative integer n, and the elements are written as n-tuples $(a_1, \ldots, a_n)$.

$|A|$ denotes the cardinal of a set A.

If m, n are integers then $[m, n]$ denotes the set of integers i such that $m \leqslant i \leqslant n$.

If α, γ are ordinals then $[\alpha, \gamma]$ and $[\alpha, \gamma)$ respectively denote the set of ordinals β with $\alpha \leqslant \beta \leqslant \gamma$ and $\alpha \leqslant \beta < \gamma$.

If I is a set and m_i, $i \in I$, are cardinals and m is a cardinal then $m = \mathrm{HCF}_{i \in I} m_i$ means that m is the largest cardinal which divides all the m_i. In practice, m is an integer, or equivalently some m_i is an integer.

$\mathbb{N}, \mathbb{Z}^+, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{R}^n$, respectively, denote the positive integers, the non-negative integers, the integers, the rationals, the reals, the complex numbers, and Euclidean n-space.

$\mathbb{Z}_2$ denotes the set consisting of two elements 0 and 1; it performs as a set, a group, a ring, a Boolean ring, a field, and a discrete topological space.

Except where otherwise indicated, functions will be written on the left of the argument, and composed accordingly. We write $\alpha: X \to Y$ or $X \xrightarrow{\alpha} Y$ to denote a function, and $x \mapsto \alpha x$ to denote its action on elements. Here $\alpha^{-1}(y) = \{x \in X \mid \alpha x = y\}$ for any $y \in Y$.

Except where otherwise specified, groups will be written multiplicatively, and abelian groups will be written additively. If $x, y \in G$ then $[x, y]$ denotes the *commutator* $x^{-1}y^{-1}xy$; this should not be confused with the above interval notation.

$H \leqslant G$ means that H is a subgroup of G.

Rings are associative and have a 1; in all situations of interest, the 0 and 1 are distinct.

(Left or right) module actions respect the 1 of the ring.

$M \oplus N$ denotes the direct sum of two modules, and $\bigoplus_{i \in I} M_i$ denotes the direct sum of a family of modules M_i indexed by the elements i of a set I.

If R is a ring, M a right R-module, and N a left R-module, then $M \otimes_R N$ denotes the tensor product, viewed as an abelian group.

The numbering treats theorems, definitions, examples, remarks, etc. as subsections, labelled, for example, as 2.9 Remarks, in Section IV.2 in Chapter IV, and referred to as Remark 2.9 within Chapter IV, and as Remark IV.2.9 within all other chapters. The end of such a subsection is indicated by ▮.

References to the bibliography are by the author–date system, with primes to distinguish publications by the same author in the same year.