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 Gregory W. Brumfiel
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Preface

This text represents an attempt to formulate foundations and rudimentary results of a type of geometry and topology in purely algebraic terms. I feel that the approach taken here is very natural and that it is only coincidental that the point of view I advocate did not emerge fifty years ago.

The mathematics itself is most similar to elementary commutative algebra and algebraic geometry. The level of difficulty is about like that of the texts on commutative algebra by Zariski - Samuel or Atiyah-Macdonald. Although, strictly speaking, the text might be read without any previous knowledge of basic commutative algebra, essential motivation would probably be lacking. On the other hand, I see no reason why a student couldn't simultaneously read this text and some classical commutative algebra.

In the final two chapters, I assume the reader is familiar with, or can read elsewhere, basic results of Artin-Schreier theory, Krull valuation theory, and algebraic geometry. The basic algebra texts listed in the bibliography as references [63] - [68] contain more than adequate background material in the appropriate sections. The final two chapters of this text are, in fact, somewhat independent of the first six chapters. I recommend that after looking at the introduction, the reader look through Chapters VII and VIII in order to gain motivation for the foundational material of Chapters I through VI.