Cambridge University Press
978-0-521-22845-9 - Partially Ordered Rings and Semi-Algebraic Geometry
Gregory W. Brumfiel
Excerpt
More information

Introduction

It is my hope that the methods developed in this text will lead to an interesting embedding of algebraic topology in a purely algebraic category, namely, some category of partially ordered rings. At the same time, the theory provides a convenient abstract setting for the theory of real semi-algebraic sets, quite analogous to commutative algebra as a setting for modern algebraic geometry.

I might motivate the study of partially ordered rings (somewhat frivolously) as follows. One observes that the integers, together with their ordering, is an initial object for a lot of mathematics. On the one hand, consideration of order properties leads to the topology of the real line, then to Euclidean spaces, and eventually to abstract continuity and general point set topology. On the other hand, consideration of arithmetic properties leads to the abstract theory of rings, fields, ideals, and modules.

Following either route, one can go too far. Completely general topological spaces and continuous maps are uninteresting. Completely general rings and modules are uninteresting. Thus the mainstream in topology concentrates on nice spaces (for example, polyhedra and manifolds) and the mainstream in algebra concentrates on nice rings (for example, finitely generated rings over fields, subrings of the complex numbers and their homomorphic images.) The two theories seem to intersect eventually in category theory and semi-simplicial homotopy theory. The topologists put back in some algebra and the algebraists put back in some topology. On the other hand, algebraic geometry works best over an algebraically closed field and such concepts as manifolds with boundary, homotopy of maps and mapping cones, which are extremely useful to topologists, are not readily available in pure algebra. A simple observation is that such concepts are easily described by algebraic equalities and inequalities in real affine space, however. Certainly all of geometry is deeply rooted in the study of equalities and inequalities of functions on real affine space. Even in point set topology real functions and inequalities play a key role, for example, in the theory of paracompact spaces.

The real number field is a formally real, real closed field. In fact, the Artin-Schreier theory of formally real fields is precisely an abstract algebraic treatment of inequalities in field theory. The real closed fields are the analogues of algebraically closed fields-they admit no proper algebraic extensions in which inequalities still make sense.

It thus seems to me that a true understanding of the relations between algebraic geometry and topology must stem from a deeper understanding of real algebraic geometry, or, actually, semi-algebraic geometry. Moreover, real algebraic geometry should not be studied by attempting to extend classical algebraic geometry to non-algebraically closed ground fields, nor by regarding the real field as a field with an added structure of a topology. Instead, the abstract algebraic treatment of inequalities originated by Artin and Schreier should be extended from fields to (partially ordered) algebras, with real closed fields replacing the algebraically closed fields as ground fields. It is obvious that such a category of partially ordered algebras provides an abstract setting for semi-algebraic geometry (study of sets defined by finitely many real polynomial equalities and inequalities), and it seems plausible that such a category would allow a natural development of algebraic topology and homotopy theory.

It is essential that the reader understand that algebraic topology (at least the homotopy category of finite simplicial complexes and the study of reasonable functors on this category) is known to be completely independent of topology, that is, independent of limits, continuity, the infinite arithmetic of open and closed sets; even the completeness of the real numbers is irrelevant. The most highly developed reduction of homotopy theory to pure algebra is the semi-simplicial, or combinatorial, approach, developed by D. M. Kan, J. C. Moore, M. M. Postnikov, and others in the 1950's. The problem with this reduction (ignoring inefficiency) is that it seems unmotivated without first developing the point set topology of finite simplicial complexes, which, in turn, is founded on the topology of the real line. Also, differential topology seems unnatural in this setting.

My philosophy is that the derivation of homotopy theory from point set topology is an historical accident. Basically, I regard the real goal to be a mathematization of our experience, sensation, and perception of space, time, and matter. This experience is inherently finite, but involves counting, hence algebra, and order relations, hence inequalities. Our *immediate* perception of boundaries of objects, and spatial and temporal order relations justifies a more structured approach than the mathematical reduction of all experience to simply counting small or big finite sets. In fact, it seems to me that a reasonable first approximation of our perception is the set theory of sets in affine space defined by finite collections of algebraic equalities and inequalities, along with the Boolean set theoretic operations of finite unions and intersections, and differences. This sort of set theory has much in common with topology, but is fundamentally very different. Thus we will often use the language of open sets, closed sets and so on, but it is to be understood that a set in the plane like $y > x^2$ is not open because it is a union (necessarily infinite) of open balls but rather because it is the set of points where the single algebraic function $y - x^2$ is positive. In general, functions can be replaced by their graphs, hence admissible functions from one semi-algebraic set to another can be thought of as certain types of semi-algebraic subsets of the product. Thus morphisms also avoid the infinite definitions of point set topology.

According to the Hilbert Basis Theorem, any set in affine space over a field defined by algebraic equalities $f_\alpha(x_1 \dots x_n) = 0$ is already defined by a finite subset of the equations $f_\alpha = 0$. Over an ordered field, subsets of affine space defined by inequalities $g_\beta(x_1 \dots x_n) \geq 0$ as well as equalities $f_\alpha(x_1 \dots x_n) = 0$ are clearly not always representable by finitely many equalities and inequalities. It is perhaps this fact which has led to the divergence of the fields of algebra and topology.

Topology is a good example of a subject which produces answers of interest before the real problems are fully clarified. As several examples of such answers, which come up in more than one context in mathematics, I would list Lie theory, the theory of compact surfaces, the Bott periodicity theorem and K-theory, the classification of differentiable structures on spheres, the theory of cohomology operations (even cohomology theory itself), the computations of the classical group bordism rings, and the emerging classification of singularities of maps. These subjects deal with topological concepts, but also turn out to be related to problems in algebra and number theory. Thus one feels certain that it is good stuff.

On the other hand, much of geometric topology is concerned with analyzing just what pathology can and cannot occur, using infinite definitions and constructions. Thus one has space filling curves, but a very strong regularity theorem about simple closed curves in the plane. I regard this as evidence that arbitrary continuous curves tend to be uninteresting, but it is not evidence that simple closed curves are interesting. In fact, in three space one has wild embedded arcs and spheres and their classification is not regarded as a mainstream problem. Under the assumption of topological local flatness, it is known that every n-sphere in $n+1$ space bounds a topological ball. More important, however, is the question of whether a smooth n-sphere in $n+1$ space bounds a smooth ball. Provocatively, if $n \neq 3$ the answer is known to be yes, but the proof is harder than the corresponding topological theorem. If $n = 3$, it is possible that some smooth copy of S^3 in $\mathbb{R}^4$ bounds a topological ball which is not diffeomorphic or piecewise linearly equivalent to the standard D^4.

Pathology can involve morphisms, as do these examples, or absolute properties of spaces. Thus topological manifolds of dimension one and two are classified, manifolds of dimension three are known to possess unique piecewise linear and differentiable structures, but are not yet classified, while it is still unknown if manifolds of dimension four and higher can always be triangulated.

There is perhaps a widespread feeling that if attention is restricted to, say, differentiable manifolds, pathology disappears. It is known (Whitehead) that smooth manifolds admit a unique compatible piecewise linear structure and it is also known (Nash, Tognoli) that compact smooth manifolds are diffeomorphic to non-singular real algebraic varieties. Also (Milnor, Serre and many other contributors) any compact manifold admits only finitely many distinct differentiable structures. But the differentiable category allows too many morphisms. Any closed set in Euclidean space can be realized as the zeros of a smooth (C^∞) function. The studies of singularities, diffeomorphisms, flows, and foliations in recent years have produced many pathological phenomena, as well as regularity theorems under suitable hypotheses. Another area in which there are strong regularity theorems for the objects is the study of smooth compact Lie group actions on compact manifolds. The compact Lie groups are algebraic groups and Palais has extended the Nash-Tognoli theorem to an equivariant version which roughly says all compact Lie group actions on compact manifolds are algebraic, up to isomorphism.

My own interest is exactly the reverse of this tradition of seeking regularity theorems in topological situations. Instead, I advocate beginning with algebra and working toward geometry, in an attempt to discover just what geometric phenomena are realizable by finite algebraic constructions. It is not so much a question of one type of mathematics being superior to another, but simply a question of how best to understand the dividing line between algebra and topology, and I think this dividing line should be approached from both directions. The notion of inequalities is very close to this dividing line, but essentially on the algebraic side. From the algebraic point of view it is more or less clear that the real algebraic numbers are just as useful as, or even preferable to, the real numbers. I will return to this philosophy from time to time in the course of this introduction.

In any event, in this book, I first develop systematically an abstract theory of partially ordered rings. The models I have in mind are rings of real valued algebraic functions on certain semi-algebraic sets in affine space and their quotients by various allowable ideals. My axioms thus reflect properties of these models, and aside from a certain amount of curiosity, the abstract theory interests me only insofar as it contributes to the eventual goals of better understanding semi-algebraic geometry and algebraic topology.

The second half of the book is devoted to a systematic introduction to real semi-algebraic geometry via Artin-Schreier Theory and the language of partially ordered rings. For the most part, the actual results can be found in the existing literature. In particular, the papers by Dubois, Efroymson, Lang, and Stengle, referred to in the bibliography are very similar in spirit to (and, in fact, influenced greatly) my philosophy. Also, there are excellent introductory accounts of Artin-Schreier theory in the algebra texts of Lang, Jacobson, and van der Waerden.

What, then, is a partially ordered ring? Generally, the definition found in the literature is a ring, together with a subset of elements called positive such that sums and products of positive elements are positive and such that if x and $-x$ are positive, then $x = 0$. (The most efficient terminology is to call 0 positive and to refer to non-zero positives as strictly positive.) *I make the further assumption that all squares are positive.* Also, of course, all rings are commutative with unit. Given such a set of positives, a partial order relation is defined on the ring by $x \geq y$ if $x - y$ is positive. The definition is purely algebraic.

The assumption that squares are positive is justified, first, because it is true in all the examples I want to fall within the scope of the theory and, secondly, because it seems to be a very useful assumption for proving analogues of the basic results in commutative algebra.

It is easy to see that a ring admits such a partial order if and only if the following condition holds: whenever $\sum_{i=1}^{n} a_i^2 = 0$, then each $a_i^2 = 0$, $1 \leq i \leq n$. The set of all finite sums of squares is then an allowable set of positives, in fact, clearly the smallest such. Nilpotent elements are decidedly permitted by this condition.

The morphisms between partially ordered rings which are important are the order preserving ring homomorphisms. Kernels of such morphisms are called convex ideals and are characterized by the property that if a sum of positive elements belongs to the ideal, then so does each summand. This gives a category, (POR).

The category (POR) turns out to be not the best approximation to the ultimate goals. A useful subcategory is the category (PORNN), partially ordered rings with no nilpotent elements. However, nilpotent elements are actually useful, just as in modern algebraic geometry. A compromise is the intermediate category (PORCK), partially ordered rings with convex killers. The added axiom is the following condition: whenever $(\sum_{i=1}^{n} p_i)x = 0$, p_i positive, then each $p_i x = 0$, $1 \leq i \leq n$. Throughout this introduction, I will indicate advantages of (PORCK). To begin, in (POR) one can have $nx = 0$, but $x \neq 0$. Since $n = 1 + \cdots + 1$, such pathology is avoided in (PORCK). Secondly, in (PORCK) the associated primes of a convex ideal are automatically convex, as are the isolated primary components. Thirdly, a ring of polynomials in finitely many indeterminates over a ring A in (PORCK) can be regarded faithfully as a ring of A-valued functions on affine space over A. A more subtle advantage is that the defining condition for (PORCK) makes sense if x takes values in a module over A. Thus, in (PORCK), a ring is an admissible module over itself, in a certain useful sense.

In any category of partially ordered rings, only ideals which are kernels of morphisms in the category should be considered at all. Thus, in (POR) one sees the convex ideals, in (PORNN), one sees the radical convex ideals $I = \sqrt{I}$, and in (PORCK) one sees what I call absolutely convex ideals. Namely, I is absolutely convex if $(\sum_{i=1}^{n} p_i)x \in I$, p_i positive, implies $p_i x \in I$, $1 \leq i \leq n$.

Much of this first volume is concerned with the partially ordered analogues of basic results on ideals in commutative algebra. Each such result must be checked, but, as a rule, slight extensions of classical arguments work for convex ideals in partially ordered rings. As examples, we mention:

1. Every proper convex ideal is contained in a maximal convex ideal. (Note maximal convex $\neq$ convex maximal.)
2. Maximal convex ideals are prime.
3. The intersection of all prime convex ideals is the (convex) ideal of all nilpotent elements.
4. The intersection of all prime convex ideals containing a given convex ideal is the nil radical of the ideal.

These results hold uniformly in the categories (POR) and (PORCK) since a convex prime ideal Q is necessarily absolutely convex. (Proof: if $(\Sigma\, p_i)x \in Q$, then $(\Sigma\, p_i)x^2 = \Sigma\, p_i x^2 \in Q$, hence $p_i x^2 \in Q$, hence $p_i\; x \in Q$.)

5. In all the categories, residue partially ordered rings are defined as the cosets of a convex or absolutely convex ideal. The positive cosets are those containing a positive element. The usual fundamental isomorphism lemmas and correspondences between ideals in residue constructions are established.
6. Localizations are defined in all the categories and the desired basic properties established. Of importance here is the concept of a concave multiplicative set, which, when it contains a positive element also contains all larger elements. This is natural, since if one wants to invert a strictly positive function on some set, then every larger function also has no zeros, hence might as well also be inverted. Complements of prime convex ideals are concave multiplicative sets, as is $S(1)$, the shadow of 1, consisting of all elements $s \geq 1$.
7. Maximal convex ideals are characterized by the property that the associated residue ring is a semi-field. By semi-field I mean that for each non-zero a, one has $ab \geq 1$ for some b. Such elements a (semi-units) are the analogues of units since they belong to no proper convex ideal.
8. Certain categorical constructions such as fibre sums, fibre products, direct and inverse limits are carried out in the various categories.
9. The set X of prime convex ideals is given the Zariski "topology",

and basic functorial and "topological" properties established. A structure sheaf of partially ordered rings is constructed by means of localizations. The stalks of the structure sheaf are the partially ordered rings one obtains by localizing with respect to complements of prime convex ideals. The ring of global sections is generally larger than the original ring. In fact, in (PORCK), the shadow of 1, $S(1) = \{s \geq 1\}$, consists of non-zero-divisors and the global sections of the structure sheaf is the ring one obtains by localizing with respect to $S(1)$. This is reasonable, since this localization exactly inverts all elements which belong to no prime convex ideals, that is, "functions nowhere zero on X". (In (POR), $S(1)$ can have zero divisors, a ring does not even inject into its global sections, in general, and the global sections cannot be described by a simple localization.)

10. A "universal bound" is obtained for roots of a monic polynomial with coefficients in a partially ordered ring. More generally, bounds are obtained for solutions of $f(x) \leq 0$, where f is a monic polynomial of even degree. As a corollary, one obtains a going-up theorem for prime convex ideals in what I call semi-integral extensions $A \subset B$. Namely, an element $x \in B$ is semi-integral over A if $f(x) \leq 0$ for some monic polynomial of even degree, with coefficients in A. Unfortunately, the going-up theorem requires a mild hypothesis on the partial orders on A and the extension B. So the theorem does not have the same applicability to, say, the real Nullstellensatz that the classical going-up theorem has to the Nullstellensatz over algebraically closed fields.

11. A theory of associated primes and isolated primary components is worked out, which is quite satisfactory for rings in (PORCK) satisfying the ascending chain condition for convex ideals. However, there is no general decomposition of convex ideals as intersections of primary convex ideals, even for Noetherian partially ordered rings. Here, as in the going-up theorem above, the gap between the abstract category (PORCK) and the specific study of finitely generated rings

over fields begins to widen.

12. Each subset X of a partially ordered ring A belongs to a smallest convex ideal H(X) (the *hull* of X) and a smallest absolutely convex ideal AH(X) (the *absolute hull* of X). If $I \subset A$ is any ideal, then $AH(I^n)AH(I^m) \subset AH(I^{n+m})$. The corresponding property for hulls does not seem to hold. Thus in (PORCK) there is a natural graded ring associated to the ideal I, namely $\bigoplus_{n \geq 0} AH(I^n)/AH(I^{n+1}) = G(A)$. Relations between A and G(A) in (PORCK) parallel properties of ordinary associated graded rings in commutative algebra.

There are two fundamental and unavoidable reasons why partially ordered algebra is "harder" than commutative algebra and why real algebraic geometry is "harder" than algebraic geometry over algebraically closed fields. The first is that principal ideals are not convex in general. The smallest convex ideal containing a given element or set of elements is rather complicated. The second is that the statement that a given polynomial over an ordered field has roots in some ordered extension field, is a non-trivial statement, requiring specific verification. These two difficulties permeate the entire theory. The second especially is perhaps the main reason why the abstract theory of partially ordered rings is not as useful in real algebraic geometry as abstract commutative algebra is in classical algebraic geometry.

Here is a simple example which illustrates certain features of the theory. If $\mathbb{R}[X,Y,Z]$ is partially ordered as a ring of real valued functions on affine space, then the ideal (X^2+Y^2+1) has no zeros, but is not convex. (Any convex ideal containing X^2+Y^2+1 must also contain $X^2, Y^2, 1$.) Slightly more subtle is the ideal (X^2+Y^2), which has zeros but for which the co-dimension of the zero set is too big. The smallest convex ideal containing X^2+Y^2 is the ideal (X^2, Y^2, XY), consisting of all functions which vanish to second order on the Z-axis. To see this, observe that the functions $(X+Y)^2$ and $(X-Y)^2$ are positive and $(X+Y)^2+(X-Y)^2 = 2(X^2+Y^2)$. So the smallest convex ideal containing X^2+Y^2 must also contain X^2, Y^2 and $(X+Y)^2$, $(X-Y)^2$. Conversely, if a sum of positive functions is in the