1 · Traces and Euler characteristics

HYMAN BASS

Columbia University

Let A be a ring. I shall write $\mathcal{P}(A)$ for the category of finitely generated projective right A-modules and $K_0(A)$ for its Grothendieck group. When A is an algebra over some commutative ring k let $\mathcal{R}_k(A)$ denote the category of right A-modules M such that $M \in \mathcal{P}(k)$, the category of 'k representations' of A, and let $R_k(A)$ denote its Grothendieck group.

I shall be mainly concerned with $\mathcal{P}(A)$ in the case when $A = kG$, the group algebra of a group G, and particularly the case when $k = \mathbb{Z}$. This is a subject that barely exists except for some very special classes of groups G, notably finite groups and abelian groups. The following questions indicate the level of our ignorance.

1. Let G be a torsion free group.

(i) Is every $P \in \mathcal{P}(\mathbb{Z}G)$ free?

No in general but there is essentially only one example known [D], Dunwoody's trefoil module. $G = \langle x, y \mid x^2 = y^3 \rangle$ is the trefoil group, P is a relation module arising from a presentation of G, and $P \oplus \mathbb{Z}G \cong \mathbb{Z}G \oplus \mathbb{Z}G$.

(ii) Is $K_0(\mathbb{Z}G) \cong \mathbb{Z}$?

No counterexamples are known.

I mention in passing the following classical problem, which turns out to be related to the above questions in certain cases.

(iii) Is $\mathbb{Z}G$ without non trivial 0-divisors?

(iv) (Serre [S]). Suppose that G is of type (FP), i.e. there is a finite resolution

$$0 \to P_n \to \dots \to P_0 \to \mathbb{Z} \to 0$$

with each $P_i \in \mathcal{P}(\mathbb{Z}G)$. Is G then of type (FL)? I.e. can one choose all the P_i to be free?

Cambridge University Press
978-0-521-22729-2 - Homological Group Theory
Edited by C. T. C. Wall
Excerpt
More information

2. Can one, for a reasonably extensive class of groups G, describe $K_0(\mathbb{Z}G)$ or at least $K_0(\mathbb{Q}G)$ in terms of the finite subgroups of G?

A specific case: Let $G = SL_n(\mathbb{Z})$ with $n \geq 3$. If H is a finite subgroup of G and if $Q \in \mathcal{P}(\mathbb{Z}H)$, we can form the induced module $P = \mathrm{Ind}_H^G(Q) \in \mathcal{P}(\mathbb{Z}G)$. My student David Carter has produced such examples which are non free [C]. In fact he shows, for any odd prime p, that $K_0(\mathbb{Z}SL_p(\mathbb{Z}))$ contains a subgroup of order $\frac{p-1}{2}$. However no one has produced non free projective modules over torsion free subgroups of $SL_n(\mathbb{Z})$. Modules obtained by induction from finite subgroups as above always restrict to free modules over torsion free subgroups.

I propose to discuss here a rank invariant r_P of projective modules $P \in \mathcal{P}(A)$. There are essentially two ways of constructing such invariants:

- Make a base change $A \to B$ so that $P \otimes_A B$ is B-free and count a basis.
- Define $r_P = T_{P/A}(1_P)$ where $T_{P/A}$ is a 'trace function' on $\mathrm{End}_A(P)$.

The first method is that used in commutative algebra, taking for B the various localizations of A. It is sometimes used also for group algebras $A = kG$, using the augmentation $A \to k$.

The second method appears to make sense only over commutative rings, since it is only then that one classically can define the trace of an endomorphism. However Stallings and Hattori introduced a trace in the general case and it is this trace that we shall use.

Conjecture (4.5) below asserts, for any group G and $P \in \mathcal{P}(\mathbb{Z}G)$, that $r_P = r_F$ for some free module F.

The final section 7 discusses various Euler characteristics constructed from such rank functions.

Much of this presentation is a resumé of results in [B].

1. Hattori-Stallings traces (see [B], [H], [St 1])

A denotes a ring; A-modules are understood to be right A-modules and $\mathcal{P}(A)$ denotes the category of those which are finitely generated and projective.

We write

$$T = T_A : A \to T(A) = A/[A, A]$$

for the natural projection to the quotient of A by the additive group $[A, A]$ generated by all commutators $[a, b] = ab - ba$.

Let P be an A-module and $P^* = \mathrm{Hom}_A(P, A)$. If $x \in P$, $a \in A$, $f \in P^*$, we have $T(f(xa)) = T(f(x)a) = T(af(x)) = T((af)(x))$, whence an additive map $P \otimes_A P^* \to T(A)$ sending $x \otimes f$ to $T(f(x))$. On the other hand we have a canonical homomorphism

$$P \otimes_A P^* \to \mathrm{End}_A(P), \quad x \otimes f \mapsto xf : y \mapsto xf(y),$$

which is an isomorphism if and only if $P \in \mathcal{P}(A)$. In this case we view the latter as an identification and so obtain an additive map, called the trace,

$$T_P = T_{P/A} : \mathrm{End}_A(P) \to T(A),$$
$$T_P(x \otimes f) = T_A(f(x)) .$$

By a (finite) coordinate system in P, we mean a finite family (x_i, f_i) in $P \times P^*$ such that $1_P = \sum x_i \otimes f_i$, in other words such that $x = \sum_i x_i f_i(x)$ for all $x \in P$. If $u \in \mathrm{End}_A(P)$, then $u(x_i \otimes f_i) = u(x_i) \otimes f_i$, so $u = u1_P = \sum_i u(x_i) \otimes f_i$ and

$$(1) \qquad T_P(u) = T_A(\sum_i f_i(u(x_i))) .$$

If (x_i) happens to be a free basis of P then (f_i) is the dual basis of P^*, $u_{ji} = f_j(u(x_i))$ defines the matrix of u relative to (x_i), and formula (1) reads: $T_P(u) = T_A(\sum_i u_{ii})$. This shows that when A is commutative and P is free then T_P is the usual trace. We define the rank of P to be the element

$$(2) \qquad r_P = r_{P/A} = T_P(1_P) = T_A(\sum_i f_i(x_i)) \in T(A) .$$

For example

Cambridge University Press
978-0-521-22729-2 - Homological Group Theory
Edited by C. T. C. Wall
Excerpt
More information

(3) $r_{A^n} = T_A(n)$.

An A-module M is said to be of type (FP) if there is a finite resolution

(4) $0 \to P_n \to \dots \to P_0 \to M \to 0$

for some $n \geq 0$ with each $P_i \in \mathcal{P}(A)$. An endomorphism $u \in End_A(M)$ can then be lifted to an endomorphism $(u_i \in End_A(P_i))$ of the resolution (4) and we put

(5) $T_M(u) = \sum_i (-1)^i T_{P_i}(u_i)$.

This gives a well defined map

(6) $T_M : End_A(M) \to T(A)$

and we define the <u>rank</u> of M to be

(7) $r_M = T_M(1_M) = \sum_i (-1)^i r_{P_i}$.

The above trace maps (6) enjoy the following properties (see [B], [H], [St]).

(1.1) <u>Additivity.</u> Let M be an A-module, M' a submodule, and M" = M/M'. If two of M', M, M" are of type (FP) so also is the third. Suppose this is the case and that $u \in End_A(M)$ leaves M' invariant and induces $u' \in End_A(M')$ and $u'' \in End_A(M'')$. Then

$$T_M(u) = T_{M'}(u') + T_{M''}(u'') .$$

(When $M = M' \oplus M''$ this corresponds to the matrix formula

$$T_M\begin{pmatrix} u' & * \\ 0 & u'' \end{pmatrix} = T_{M'}(u') + T_{M''}(u'') .)$$

(1.2) <u>Linearity.</u> If $u, v \in End_A(M)$ then

$$T_M(u + v) = T_M(u) + T_M(v) .$$

Further, if A is a k-algebra for some commutative ring k, then so also is $End_A(M)$, T(A) is naturally a k-module, and T_M is k-linear,

$$T_M(ut) = T_M(u)t$$

for $t \in k$. Consequently r_M is annihilated by $ann_k(M)$.

(1.3) Commutativity. Let $M \underset{u'}{\overset{u}{\rightleftarrows}} M'$ be homomorphisms between A-modules of type (FP). Then

$$T_M(u'u) = T_{M'}(uu').$$

(1.4) Universality. Suppose $T'_P : End_A(P) \to S$ is a collection of maps (defined for $P \in \mathcal{P}(A)$) into an additive group S, which is additive, linear, and commutative in the above sense. Then there is a unique group homomorphism $t : T(A) \to S$ such that $T'_P = t \circ T_P$ for all $P \in \mathcal{P}(A)$. The same applies if we replace $\mathcal{P}(A)$ by the category of all A-modules of type (FP).

(1.5) Functoriality; covariance. A ring homomorphism $\alpha : A \to B$ induces an additive map

$$\alpha_* : T(A) \to T(B) ,$$
$$T_A(a) \mapsto T_B(\alpha(a)) .$$

If $P \in \mathcal{P}(A)$ and $u \in End_A(P)$ then we have $\alpha_* P = P \otimes_A B \in \mathcal{P}(B)$ and $\alpha_* u = u \otimes_A 1_B \in End_B(\alpha_* P)$, and

$$T_{\alpha_* P}(\alpha_* u) = \alpha_* T_P(u) .$$

In particular $r_{\alpha_* P} = \alpha_* r_P$. If B is a flat left A-module (via α), then these formulae remain valid for all A-modules P of type (FP).

(1.6) Automorphisms. Suppose that α is an automorphism of A. For every A-module M we have the A-module $M^{(\alpha)}$ with M as additive group and scalar operation $x.a = x\alpha(a)$. Then $M \mapsto M^{(\alpha)}$, and $u \mapsto u^{(\alpha)} = u$ for morphisms, is an automorphism of the category of A-modules. The map $x \mapsto x \otimes 1$ is an A-isomorphism from $M^{(\alpha)}$ to

$\alpha_*^{-1}M = M \otimes_{\alpha^{-1}} A$, matching $u \in \mathrm{End}_A(M) = \mathrm{End}_A(M^{(\alpha)})$ with $\alpha_*^{-1}u = u \otimes_{\alpha^{-1}} 1_A \in \mathrm{End}_A(\alpha_*^{-1}M)$. It follows therefore from (1.5) that if M is of type (FP) then

$$T_{M^{(\alpha)}}(u) = \alpha_*^{-1}T_M(u) .$$

In particular

$$r_{M^{(\alpha)}} = \alpha_*^{-1}r_M .$$

(1.7) Contravariance; $\mathrm{Tr}_{B/A}$. If α makes B a right A-module of type (FP) then it does the same to all B-modules M of type (FP), so we can define $T_{M/A}(u)$ for $u \in \mathrm{End}_B(M) \subset \mathrm{End}_A(M)$. The map $(M, u) \mapsto T_{M/A}(u)$ is manifestly additive, linear, and commutative. By universality, therefore, it is of the form

$$T_{M/A}(u) = \mathrm{Tr}_{B/A}(T_{M/B}(u))$$

for a unique homomorphism

$$\mathrm{Tr}_{B/A} : T(B) \to T(A) .$$

2. Characters

Let k be a commutative ring and let A be a k-algebra. If M is an A-module and $a \in A$, the endomorphism $a_M : x \mapsto xa$ of M is k-linear. Let $\mathcal{R}_k(A)$ denote the category of A-modules M which are finitely generated and projective as k-modules. If $M \in \mathcal{R}_k(A)$ we have its character

$$\chi_M : A \to k$$
$$a \mapsto T_{M/k}(a_M) .$$

It is a k-linear map vanishing on $[A, A]$, so we may also view χ_M as an element of $\mathrm{Hom}_k(T(A), k)$.

The additive functor $H : P \mapsto \mathrm{Hom}_A(P, M)$ sends $\mathcal{P}(A)$ to $\mathcal{P}(k)$. If $P \in \mathcal{P}(A)$ and $u \in \mathrm{End}_A(P)$, we have $T_{H(P)/k}(H(u)) \in k$, which is clearly additive, linear, and commutative in (P, u), whence a homomorphism $\chi : T(A) \to k$ such that $T_{H(P)/k}(H(u)) = \chi(T_{P/A}(u))$. When $P = A$ and $u(x) = ax$ we have an isomorphism $(H(P). H(u)) \cong (M, a_M)$, whence $\chi = \chi_M$. Explicitly,

(2.1) **Proposition.** <u>If</u> $M \in \mathcal{R}_k(A)$, $P \in \mathcal{P}(A)$, <u>and</u> $u \in \mathrm{End}_A(P)$ <u>then</u> $\mathrm{Hom}_A(P, M) \in \mathcal{P}(k)$ <u>and</u>

$$T_{\mathrm{Hom}_A(P,M)/k}(\mathrm{Hom}_A(u, M)) = \chi_M(T_{P/A}(u)).$$

<u>In particular when</u> $u = 1_P$ <u>we have</u>

$$r_{\mathrm{Hom}_A(P, M)/k} = \chi_M(r_{P/A}) .$$

(2.2) **Proposition.** <u>Suppose that</u> A <u>is a finitely generated projective</u> k-<u>module. Then every</u> $P \in \mathcal{P}(A)$ <u>is likewise and, if</u> $r_{P/A} = T_A(a)$, <u>we have</u>

$$\chi_P(b) = T_{A/k}(L_a \circ R_b) = T_{A/k}(x \mapsto axb) .$$

In fact let a_i, $f_i : A \to k$ be a finite k-coordinate system of A and let x_j, $g_j : P \to A$ be a finite A-coordinate system of P. If $x \in P$ then $\sum_{i,j} x_j a_i f_i(g_j(x)) = \sum_j x_j g_j(x) = x$, so $x_j a_i$, $f_i g_j : P \to k$ is a k-coordinate system of P. Hence $\chi_P(b) = T_{P/k}(b_P) = \sum_{i,j} f_i g_j(b_P(x_j a_i)) = \sum_{i,j} f_i(g_j(x_j a_i b)) = \sum_i f_i(aa_i b)$ (where $a = \sum_j g_j(x_j)$), so $r_P = T_A(a)$

$$= T_{A/k}(x \mapsto axb).$$

3. Group algebras

Let $A = kG$, the group algebra of a group G over a commutative ring k. The k-module $[A, A]$ is generated by the commutators

$$[s, t] = st - ts = sus^{-1} - u = [su, s^{-1}]$$

where $s, t, u \in G$ and $u = ts$. Thus $T(s) = T(t)$ in $T(kG)$ if and only if s and t are conjugate in G. We shall thus identify $T(s)$ (or $T_G(s)$) with the G-conjugacy class of s. These classes constitute a k-basis

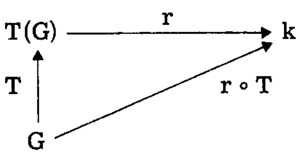
CAMBRIDGE

Cambridge University Press
978-0-521-22729-2 - Homological Group Theory
Edited by C. T. C. Wall
Excerpt
More information

T(G) of T(kG). If $r \in T(kG)$ we thus have

$$r = \sum_{\tau \in T(G)} r(\tau) \cdot \tau,$$

a notation that interprets r as a function



with finite support $\mathrm{supp}(r) = \{\tau \in T(G) \mid r(\tau) \neq 0\}$. We shall sometimes confuse r with the central function $r \circ T$, writing $r(s)$ for $r(T(s))$ if $s \in G$. For any function f on G we define $\bar{f}$ by $\bar{f}(s) = f(s^{-1})$.

(3.1) Proposition (Hattori [H]). *Let* G *be a finite group and let* $P \in \mathcal{P}(kG)$. *Then*

$$\chi_P(s) = |Z_G(s)| \cdot r_P(s^{-1})$$

for $s \in G$. *For* $s = 1$ *this gives*

$$r_{P/k} = |G| \cdot r_P(1) .$$

In fact let $a = \sum_{s \in G} a_s s \in kG$ be such that $r_P = T(a)$. According to Proposition (2.2) we have $\chi_P(s) = T_{kG/k}(x \mapsto axs) = \sum_{t \in G} a_t T_{kG/k}(x \mapsto txs)$. Now $x \mapsto txs$ permutes the k-basis G of kG so its trace is the number of $x \in G$ such that $txs = x$, i.e. such that $t = xs^{-1}x^{-1}$. This number is 0 if $t \notin T(s^{-1})$, and $|Z_G(s)|$ if $t \in T(s^{-1})$. Thus $\chi_P(s) = \sum_{t \in T(s^{-1})} a_t \cdot |Z_G(s)| = r_P(s^{-1}) \cdot |Z_G(s)|$.

(3.2) Corollary. *If* $|G|$ *is invertible in* k *then* $r_P = T(a_P)$ *where* $a_P = |G|^{-1} \sum_{s \in G} \chi_P(s^{-1})s$, *an element of the center of* kG.

(3.3) Corollary. *If* k *is an integral domain in which no prime divisor of* $|G|$ *is invertible then* 0 *and* 1 *are the only idempotents in* kG.

If $\mathrm{char}(k) = p > 0$ then G is a p-group so, if F is the field of fractions of k, then FG is a local ring; whence the corollary.

Suppose $p = 0$ and let $e \neq 0$ be an idempotent in kG. Put $P = ekG$. Then $r_{P/k} = \chi_P(1) = |G| \cdot r_P(1)$, so $r_{P/k}/|G|$ belongs to $k \cap \mathbb{Q}$ and our hypothesis implies that it is an integer. But if $|G|$ divides $r_{P/k}$ and P is a direct summand of kG we must have $P = kG$, so $e = 1$.

4. Subgroups of finite index

Let H be a subgroup of finite index in G. Then

$$kG = \bigoplus_{s \in G/H} skH ,$$

a free kH-module with basis a set of representatives of the cosets G/H. Therefore we have a k-linear map (see (1. 7))

$$Tr = Tr_{kG/kH} : T(kG) \to T(kH)$$

defined by $Tr(T_G(a)) = T_{kG/kH}(L_a : x \mapsto ax)$ for $a \in kG$. If $t \in G$ then L_t permutes the direct summands skH above, and $tskH = skH$ if and only if $s^{-1}ts \in H$, in which case $L_t(s) = s \cdot (s^{-1}ts)$. Therefore

(1) $$Tr(T_G(t)) = \sum_{\substack{s \in G/H \\ s^{-1}ts \in H}} T_H(s^{-1}ts).$$

Let $\tau = T_G(t)$. Then (1) shows that

(2) $$Tr(\tau) = \sum_{\substack{\sigma \in T(H) \\ \sigma \subset \tau}} z_\sigma \cdot \sigma$$

where z_σ is the number of $s \in G/H$ such that $s^{-1}ts \in \sigma$. If s_0 has this property then s_1 does also if and only if $s_1 \in Z_G(t)s_0H$, so z_σ is the number of H-cosets in the double coset $Z_G(t)s_0H$. This is the index in $Z_G(t)$ of $Z_G(t) \cap s_0Hs_0^{-1} = Z_{s_0Hs_0^{-1}}(t)$, so

(3) $$z_\sigma = [Z_G(s) : Z_H(s)]$$

for any $s = s_0^{-1}ts_0 \in \sigma$. Suppose that $r = \sum_{\tau \in T(G)} r(\tau)\tau \in T(kG)$. Then

$\mathrm{Tr}(r) = \sum_{\tau\in T(G)} r(\tau) \sum_{\substack{\sigma\in T(H)\\ \sigma\subset\tau}} z_\sigma \cdot \sigma = \sum_{\sigma=T_H(s)\in T(H)} r(s) \cdot z_\sigma \sigma$. In other words, for $s \in H$ we have

(4) $\quad \mathrm{Tr}(r)(s) = r(s) \, . \, [Z_G(s) : Z_H(s)]$.

If M is a kG-module of type (FP), hence likewise as kH-module, and if $u \in \mathrm{End}_{kG}(M)$, we have $T_{M/kH}(u) = \mathrm{Tr}(T_{M/kG}(u))$. In case $u = 1_M$ we thus have from (4):

(5) $\quad r_{M/H}(s) = r_{M/G}(s) \, . \, [Z_G(s) : Z_H(s)]$

for $s \in H$. When $s = 1$ this becomes

(6) $\quad r_{M/H}(1) = r_{M/G}(1) \, . \, [G : H]$.

(4.1) Theorem. Let G be a finite group and let k be an integral domain in which no prime divisor of $|G|$ is invertible. Let $P \in \mathcal{P}(kG)$ and let n denote the rank of the k-module $P \otimes_{kG} k$. Then

(7) $\quad r_P = r_{(kG)^n} \quad (= T_{kG}(n))$.

Since $n = \sum_{\tau\in T(G)} r(\tau)$, the theorem is equivalent to the assertion that

(8) $\quad r_P(s) = 0$ for $s \neq 1$ in G.

Let $s \in G$ and put $H = \langle s \rangle$. Then $r_{P/H}(s) = r_{P/G}(s) \, . \, [Z_G(s) : Z_H(s)]$, and the last factor is $\neq 0$ in k, by assumption. Thus it suffices to prove the theorem for the abelian group H. But then $r_{P/H}$ lies in the subring of kH generated by all idempotents. By Corollary (3.3) above, this subring is the prime subring, whence $r_{P/H}(s) = 0$ if $s \neq 1$.

(4.2) Corollary (Swan). Let F be the field of fractions of k. Then $P \otimes_k F \cong (FG)^n$.

If $\mathrm{char}(k) = p > 0$ then G is a p-group, so FG is a local ring, and $P \otimes_k F$ is FG-free. If $p = 0$ then FG is semi-simple and it suffices to show that $\chi_P = \chi_{(kG)^n}$. In view of Proposition (3.1), this