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1 · Choquet theory

Here the basic ideas of Choquet theory are developed in a framework suitable for uniform algebras. The lectures of R. Phelps[6] provide a very readable account of Choquet theory, as does also the expository paper of G.Choquet and P.A. Meyer[3]. Their approach has been modified by D.A.Edwards[4], in order to handle Jensen measures and the Jensen-Hartogs inequality for function algebras. We will follow the development of Choquet and Meyer, as amended by Edwards.

$\mathcal{R}$-measures

Let M be a compact space, and let $\mathcal{R}$ be a family of continuous functions from M to the extended line $[-\infty,+\infty)$. We will assume always that $\mathcal{R}$ has the following properties.

$$\mathcal{R} \text{ includes the constant functions.} \tag{1.1}$$

$$\text{If } m \in Z_+ \text{ and } v,w \in \mathcal{R}, \text{ then } (v+w)/m \in \mathcal{R}. \tag{1.2}$$

$$\mathcal{R} \text{ separates the points of } M. \tag{1.3}$$

An *$\mathcal{R}$-measure* for $\phi \in M$ is a probability measure σ on M such that

$$w(\phi) \le \int w\,d\sigma, \quad w \in \mathcal{R}. \tag{1.4}$$

Since $\mathcal{R}$ includes the constants, the estimate (1.4) is equivalent to the estimate

$$\int w\,d\sigma \ge 0, \text{ for all } w \in \mathcal{R} \text{ such that } w(\phi) = 0. \tag{1.5}$$

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As an example, observe that the point mass δ_ϕ at ϕ is always an R-measure for ϕ .

The theory applies to any linear subset R of $C_R(M)$ that contains the constants and separates the points of M . In this case, the fact that $R = -R$ implies that the R-measures for a point $\phi \in M$ are the representing measures for ϕ , that is, the probability measures σ on M that satisfy

$$u(\phi) = \int u d\sigma , \quad \text{all} \quad u \in R .$$

In the principal application dealt with by Choquet theory, M is a compact convex subset of a locally convex linear topological vector space, and R is the space of continuous real-valued affine functions on M . In this case, each probability measure σ on M is an R-measure for some $\phi \in M$, ϕ being referred to as the "barycenter" of σ .

The main example that will occupy our attention is the case in which R consists of functions of the form $(\log |f|)/m$, where m is a positive integer, and f belongs to an algebra A of continuous complex-valued functions on some compact space M . In this case, a probability measure σ on M is an R-measure for $\phi \in M$ if and only if the Jensen-Hartogs inequality is valid:

$$\log |f(\phi)| \leq \int \log |f| \, d\sigma , \quad f \in A .$$

The R-measures are called *Jensen measures*.

Now fix $\phi \in M$, and fix a compact subset X of M . Let U be the set of functions $u \in C_R(X)$ such that $u > w$ for some $w \in R$ satisfying $w(\phi) = 0$. If the functions in R are continuous, then U is simply the algebraic sum of the positive continuous functions on X , and the functions in R vanishing at ϕ .

On account of (1.2), $\mathcal{U}$ is a convex cone. Since $0 \in \mathcal{R}$, the cone $\mathcal{U}$ includes the positive functions in $C_R(X)$.

Since the restriction of every $w \in \mathcal{R}$ to X is a lower envelope of functions in $\mathcal{U}$, a probability measure σ on X is an $\mathcal{R}$-measure if and only if $\int u\,d\sigma \geq 0$ for all $u \in \mathcal{U}$. In particular, the $\mathcal{R}$-measures can be described by inequalities involving integrals of continuous functions, so that the set of $\mathcal{R}$-measures on X is a convex, weak-star compact set.

1.1 Lemma. Let X be a compact subset of M, and let $\phi \in M$. There exists an $\mathcal{R}$-measure σ for ϕ supported on X if and only if

$$w(\phi) \leq \sup_{x \in X} w(x), \quad w \in \mathcal{R}. \tag{1.6}$$

Proof. If there is an $\mathcal{R}$-measure σ for ϕ on X, then the inequality $w(\phi) \leq \int_X w\,d\sigma$ yields (1.6) immediately. Conversely, if (1.6) is valid, then the constant function -1 does not belong to $\mathcal{U}$, and $\mathcal{U}$ is a proper cone in $C_R(X)$. By the separation theorem for convex sets, there is a nonzero measure τ on X such that $\int u\,d\tau \geq 0$ for all $u \in \mathcal{U}$. Since $\mathcal{U}$ includes the positive functions, τ is a positive measure. The measure $\sigma = \tau/\tau(X)$ is then a probability measure that is nonnegative on $\mathcal{U}$, so that σ is an $\mathcal{R}$-measure. □

The following version of the monotone extension theorem is due in this setting to D.A.Edwards[4].

1.2 Theorem (Edwards' Theorem). Let $\phi \in M$, and let X be a compact subset of M. For each lower semi-continuous function Q from X to $(-\infty, +\infty]$, the following quantities are equal:

$$\sup\{w(\phi) : w \in \mathcal{R},\ w \le Q \text{ on } X\}, \tag{1.7}$$

$$\inf\{\textstyle\int Q\,d\sigma : \sigma \text{ is an } \mathcal{R}\text{-measure for } \phi \text{ on } X\}. \tag{1.8}$$

Here the infimum in (1.8) is declared to be $+\infty$ if there is no $\mathcal{R}$-measure for ϕ on X.

Proof. Let S denote the supremum in (1.7) and let I denote the infimum in (1.8). If there are no $\mathcal{R}$-measures for ϕ on X, there is by Lemma 1.1 a function $w \in \mathcal{R}$ such that $w(\phi) > 0$, while $w < 0$ on X. Multiplying w by a large positive constant, we can arrange that $w \le Q$ on X, while $w(\phi)$ is arbitrarily large. Hence $S = +\infty$, so that $S = I$.

We can assume then that there exist $\mathcal{R}$-measures for ϕ on X. If σ is such an $\mathcal{R}$-measure, and if $w \in \mathcal{R}$ satisfies $w \le Q$, then $w(\phi) \le \int w\,d\sigma \le \int Q\,d\sigma$. Hence $S \le I$.

To prove the reverse inequality, suppose first that Q is continuous. Let $b > S$. Then there is no $w \in \mathcal{R}$ such that $w(\phi) = 0$ and $w + b \le Q$. Consequently $Q - b$ does not belong to the cone $\mathcal{U}$ defined earlier. By the separation theorem for convex sets, there is a nonzero measure τ on X such that τ is nonnegative on $\mathcal{U}$, while $\int(Q-b)d\tau \le 0$. Since $\tau \ge 0$ on $\mathcal{U}$, $\sigma = \tau/\tau(X)$ is an $\mathcal{R}$-measure. Furthermore, $\int(Q-b)d\sigma \le 0$, so that $\int Q\,d\sigma \le b$, and $I \le b$. Since $b > S$ is arbitrary, we conclude that $I = S$, in the case at hand.

For the general case, consider a continuous function $q \le Q$, and let $I_q = S_q$ denote the quantity above determined by q. For each such q, choose an $\mathcal{R}$-measure σ_q such that $\int q\,d\sigma_q = S_q$. This choice is possible; since the set of $\mathcal{R}$-measures is weak-star compact. Let σ be a weak-star adherent point of the net $\{\sigma_q\}_{q<Q}$ as q increases to Q. If q is fixed, and the continuous function p satisfies $q \le p \le Q$, then $\int q\,d\sigma_p \le \int p\,d\sigma_p = S_p \le S$. Letting p

increase to Q, we obtain $\int q\,d\sigma \le S$. Since $q \le Q$ is arbitrary, $\int Q\,d\sigma \le S$. Hence $I = S$, and moreover there exists an R-measure σ such that $\int Q\,d\sigma = I$. □

Applying Edwards' Theorem to the function $-\chi_E$, where χ_E is the characteristic function of a closed subset E of X, we obtain the following corollary.

1.3 Corollary. Suppose there is an R-measure on X for ϕ. Then for any closed subset E of X,

$$\sup\{\sigma(E) : \sigma \text{ an } R\text{-measure on } X \text{ for } \phi\}$$

is equal to

$$\inf\{u(\phi) : u \in -R,\ u \ge 0 \text{ on } X,\ u \ge 1 \text{ on } E\}.$$

The Family S of R-envelope Functions

A lower semi-continuous function u from a compact subset X of M to $(-\infty,+\infty]$ is an R-*envelope function* on X if u is the upper envelope on X of functions in R. The family of R-envelope functions on M is denoted by S. In the abstract framework, the R-envelope functions often play the role that the subharmonic functions play in classical potential theory.

Every R-envelope function on X is evidently the restriction to X of a function in S.

If $c \in \mathbb{R}$, and if $w_1,\ldots,w_n \in R$, then the function

$$\max(c,w_1,\ldots,w_n) \tag{1.9}$$

is a continuous R-envelope function. The functions in S are simply the upper envelopes of functions of the form (1.9). An elementary compactness argument establishes the following.

1.4 Lemma. The continuous $\mathcal{R}$-envelope functions are the uniform limits of functions of the form $\max(c, w_1, \ldots, w_n)$, where c is real and $w_1, \ldots, w_n \in \mathcal{R}$.

From the definitions, we see immediately that $\mathcal{S}$ has the following properties.

$\mathcal{S}$ includes the constant functions. (1.10)

If $u \in \mathcal{S}$ and $c > 0$, then $cu \in \mathcal{S}$. (1.11)

If $u, v \in \mathcal{S}$, then $u + v \in \mathcal{S}$. (1.12)

If $\{u_\alpha\}$ is any subset of $\mathcal{S}$, then $\sup_\alpha u_\alpha$ belongs to $\mathcal{S}$. (1.13)

The following property of $\mathcal{S}$ is sufficiently important to merit a separate statement.

1.5 Lemma. If $u \in \mathcal{S}$, and if χ is an increasing continuous convex function from an interval containing the range of u to $(-\infty, +\infty]$, then $\chi \circ u \in \mathcal{S}$.

Proof. There is only one possible point of discontinuity of an arbitrary increasing convex function χ , namely, a point β such that $\chi(t) = +\infty$ for $t > \beta$, while $\chi(t) < +\infty$ for $t < \beta$. We must assume that $\chi(t)$ tends to $\chi(\beta)$ as t increases to β . In this case, χ is an upper envelope of functions of the form $at + b$, where $a > 0$. Consequently $\chi \circ u$ is an upper envelope of functions of the form $au + b$, where $a > 0$. Since each of these belongs to $\mathcal{S}$, so does $\chi \circ u$. □

The $\mathcal{R}$-envelope functions are dual, in some sense, to $\mathcal{R}$-measures. This duality is exhibited by the following characterization of the $\mathcal{R}$-envelope functions.

1.6 Theorem. Let u be a lower semi-continuous function from M to $(-\infty,+\infty]$. Then u is an $\mathcal{R}$-envelope function if and only if

$$u(\phi) \le \int u\,d\sigma \tag{1.14}$$

for all $\phi \in M$ and all $\mathcal{R}$-measures σ on M for ϕ.

Proof. Since (1.14) holds for all $u \in \mathcal{R}$, it also holds for all upper envelopes of functions in $\mathcal{R}$, hence for all $u \in \mathcal{S}$.

Conversely, suppose that (1.14) is valid for all $\phi \in M$ and all $\mathcal{R}$-measures σ on M for ϕ. Let v be any continuous function on M such that $v < u$. According to Edwards' Theorem (Theorem 1.2), there exists for each $\phi \in M$ and each $\varepsilon > 0$, a function $w \in \mathcal{R}$ such that $w < u$, while $w(\phi) > u(\phi)-\varepsilon$. It follows that u is an upper envelope of functions in $\mathcal{R}$. □

From Theorem 1.6 and Fatou's Lemma, we obtain immediately the following.

1.7 Corollary. If $\{u_j\}_{j=1}^{\infty}$ is a sequence in $\mathcal{S}$ that is bounded above, and if $u = \limsup_{j\to\infty} u_j$ is bounded and lower semi-continuous, then $u \in \mathcal{S}$.

There is another simple proof of Lemma 1.5, based on Theorem 1.6 and Jensen's inequality. Recall that *Jensen's inequality* is the estimate

$$\chi\left(\int u\,d\sigma\right) \le \int \chi\circ u\; d\sigma , \tag{1.15}$$

valid whenever σ is a probability measure, u is real-valued and χ is an increasing convex real-valued function of a real variable. The validity of (1.15) for simple functions boils down to the convexity of χ.

To prove Lemma 1.5, one notes that if σ is an R-measure for ϕ, then $\chi(u(\phi)) \le \chi(\int u d\sigma) \le \int \chi\circ u \, d\sigma$, so that by Theorem 1.6, $\chi\circ u \in S$.

The Family S_C of Continuous R-envelope Functions

We denote by S_C the family of (finite) R-envelope functions on M that are continuous. As observed earlier in Lemma 1.4, these are the uniform limits of the functions of the form (1.9).

Evidently S_C is a convex cone that separates points and contains the constants. Consequently S_C enjoys the properties (1.1), (1.2) and (1.3) postulated for R. The theory we have developed can be applied to S_C in place of R. Observe though that the S_C-measures are precisely the R-measures, while the S_C-envelope functions coincide with the R-envelope functions. For many purposes, the family R can be replaced by the family S_C.

An important property enjoyed by S_C, but not necessarily by R, is that of being a semi-lattice. The maximum of any two functions in S_C again belongs to S_C. This leads to the following observation, which plays a crucial role in the treatment of maximal measures.

1.8 Lemma. The algebraic difference $S_C - S_C$ is dense in $C_R(M)$.

Proof. If $v_1, v_2, w_1, w_2 \in S_C$, then

$$\max(v_1 - w_1, v_2 - w_2) = \max(v_1 + w_2, v_2 + w_1) - w_1 - w_2 .$$

It follows that $S_C - S_C$ is a lattice. Since it separates points and contains the constants, it is dense in $C_R(M)$, by the lattice version of the Stone-Weierstrass Theorem. □

The $\mathcal{R}$-Dirichlet Problem

Let u be a lower semi-continuous function from M to $(-\infty,+\infty]$. In analogy to the procedure followed by Perron to solve the Dirichlet problem, we define the (lower) *solution to the $\mathcal{R}$-Dirichlet problem* with data u on M to be the upper envelope $\tilde{u}$ of the functions in $\mathcal{R}$ dominated by u on M:

$$\tilde{u}(\phi) = \sup\{w(\phi) : w \in \mathcal{R}, w < u \text{ on } M\}. \tag{1.16}$$

The supremum defining $\tilde{u}$ could as well be taken over the functions in $\mathcal{S}$, or in $\mathcal{S}_C$, dominated by u. Edwards' Theorem gives an alternative expression for $\tilde{u}$:

$$\tilde{u}(\phi) = \inf\{\int u\,d\sigma : \sigma \text{ an } \mathcal{R}\text{-measure on } M \text{ for } \phi\}.$$

Some elementary properties of the correspondence $u \to \tilde{u}$ are as follows:

$$\tilde{u} \text{ is an } \mathcal{R}\text{-envelope function}, \tag{1.17}$$

$$\tilde{u} = u \text{ if and only if } u \in \mathcal{S}, \tag{1.18}$$

$$\widetilde{cu} = c\tilde{u} \text{ if } c > 0, \tag{1.19}$$

$$\tilde{u} + \tilde{v} \le \widetilde{u+v}, \tag{1.20}$$

$$\tilde{u} \le \tilde{v} \text{ whenever } u \le v, \tag{1.21}$$

if a net $\{u_\alpha\}$ of lower semi-continuous functions increases pointwise on M to u, then $\tilde{u}_\alpha$ increases pointwise on M to $\tilde{u}$. (1.22)

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In applications, we will wish to consider a lower semi-continuous boundary function u defined only on a compact subset X of M. Again $\tilde{u}$ is defined to be the upper envelope on M of the functions in $\mathcal{S}$ dominated by u on X. This amounts to declaring u to be $+\infty$ on $M \backslash X$, and defined $\tilde{u}$ as before. From Edwards' Theorem, we obtain

$$\tilde{u}(\phi) = \inf\{\int u d\sigma : \sigma \text{ an } \mathcal{R}\text{-measure on } X \text{ for } \phi\}, \quad (1.23)$$

where the infimum is declared to be $+\infty$ if there are no $\mathcal{R}$-measures for ϕ on X.

The Choquet Boundary

The *Choquet boundary* of $\mathcal{R}$, denoted by $\partial_{\mathcal{R}}$, consists of those points $\phi \in M$ such that the point mass δ_ϕ at ϕ is the only $\mathcal{R}$-measure for ϕ.

By Edwards' Theorem, any continuous real-valued function u on M satisfies $\tilde{u} = u$ on the Choquet boundary. This property characterizes the Choquet boundary. Indeed, suppose ϕ is not a Choquet boundary point, and choose an $\mathcal{R}$-measure σ for ϕ such that $\sigma \neq \delta_\phi$. Then any $u \in C_R(M)$ satisfying $\int u d\sigma < u(\phi)$, also satisfies $\tilde{u}(\phi) < u(\phi)$.

The next lemma shows that the notion of Choquet boundary point is independent, in the appropriate sense, of the compact set on which the functions in $\mathcal{R}$ are defined.

1.9 Lemma. Let X be a closed subset of M such that every $w \in \mathcal{R}$ attains its maximum on X, and let $x_0 \in X$. If the point mass at x_0 is the only $\mathcal{R}$-measure for x_0 on X, then x_0 belongs to the Choquet boundary of $\mathcal{R}$.

Proof. Let σ be an $\mathcal{R}$-measure for x_0 on M. Let E be a compact neighbourhood of x_0 in X, and let u be a continuous function on X such that $u \leq 0$, $u(x_0) = 0$, and