Part I · Crossed products of von Neumann algebras

1. INTRODUCTION

A covariant system is a triple (M, G, α) where M is a von Neumann algebra, G is a locally compact group and α is a continuous action of G on M, that is a homomorphism $\alpha : s \to \alpha_s$ of G into the group of *-automorphisms of M such that for each $x \in M$, the map $s \to \alpha_s(x)$ is continuous from G to M where M is considered with its strong topology. To such a covariant system is associated in a natural way a new von Neumann algebra, called the crossed product of M by the action α of G, and denoted here by $M \otimes_\alpha G$ [16].

Similarly also covariant systems over C*-algebras are defined. In fact they have been known by mathematical physicists for some time already (see e.g. [7]). There they arise naturally because of time evolution of the physical system.

Here we will only be concerned with covariant systems over von Neumann algebras. Also they arise in a quite natural way, indeed the Tomita-Takesaki theory associates a strongly continuous one-parameter group of automorphisms to each faithful normal state on a von Neumann algebra [13, 15, 17], and clearly such a group is nothing else but a continuous action of **R**.

The crossed product construction can be used to provide new, more complicated examples of von Neumann algebras. A special case of this, the group measure space construction, was already used by Murray and von Neumann to obtain non type I factors [6]. Recently also Connes used the crossed product construction to obtain an example of a von Neumann algebra, not anti-isomorphic to itself [3].

In connection with the Tomita-Takesaki theory however crossed products are also used to obtain structure theorems for certain types of von Neumann algebras. Among those we have the results of Connes about the structure of III_λ-factors with $0 \leq \lambda < 1$ [2] and the result of Takesaki which states that every type III von Neumann algebra is isomorphic to the crossed product of a type II_∞ von Neumann algebra with some continuous action of $\mathbf{R}$ [16].

In part II of these lecture notes we treat Takesaki's structure theorem. In this part we deal with the notion of general continuous crossed products. This notion is introduced in section 2. If (M, G, α) is a covariant system, and if M acts in the Hilbert space $\mathcal{H}$, then the crossed product $M \otimes_\alpha G$ will act in the space $\mathcal{H} \otimes L_2(G)$ where $L_2(G)$ is the Hilbert space of square integrable functions on G with respect to some left Haar measure.

In section 3 we define an action θ of G on $M \otimes \mathcal{B}(L_2(G))$ where $\mathcal{B}(L_2(G))$ is the von Neumann algebra of all bounded operators on $L_2(G)$ and we show that $M \otimes_\alpha G$ can be characterized as the fixed points in $M \otimes \mathcal{B}(L_2(G))$ for the automorphisms $\{\theta_t \mid t \in G\}$. This result was obtained by Takesaki [16] in a special case, and by Digernes [4, 5] and Haagerup [8] in more general cases using the theory of dual weights. If G is compact the result can easily be obtained using the normal projection map $\int \theta_t \, dt$ onto the fixed points, where dt is the normalized Haar measure on G. In our approach we use a carefully chosen approximation procedure to make a similar argument work also if G is not compact (see also [18]). If the action α is spatial, an expression of the commutant of $M \otimes_\alpha G$ follows.

Finally in section 4 we consider the abelian case. If G is abelian it is possible to associate to a covariant system (M, G, α) a new covariant system $(\hat{M}, \hat{G}, \hat{\alpha})$ in a canonical way. For $\hat{M}$ one takes $M \otimes_\alpha G$, and $\hat{\alpha}$ is a continuous action of the dual group $\hat{G}$ on $M \otimes_\alpha G$. The action $\hat{\alpha}$ is defined in such a way that the crossed product $(M \otimes_\alpha G) \otimes_{\hat{\alpha}} \hat{G}$ of $M \otimes_\alpha G$ by the action $\hat{\alpha}$ of $\hat{G}$ is isomorphic to $M \otimes \mathcal{B}(L_2(G))$. This makes a duality structure possible if M is properly infinite and G separable so that $M \otimes \mathcal{B}(L_2(G))$ is isomorphic to M again. Then the covariant system canonically associated to $(\hat{M}, \hat{G}, \hat{\alpha})$ is in some sense equivalent to the original one (M, G, α) [16]. Our method here is very much dependent on operators similar to the unitary U in $L_2(G \times G)$ defined by $(Uf)(s, t) = f(ts, t)$. They enable us to work with tensor products and the structure becomes simpler and more transparent.

For the theory of von Neumann algebras we refer to the books of Dixmier [6] and Sakai [14]. For the Haar measure we refer to [1]. In the case where the groups are σ-compact, in particular if $G = \mathbf{R}$ as in part II, many other books treating Haar measure and abstract harmonic analysis will do [9, 11].

2. CROSSED PRODUCTS OF VON NEUMANN ALGEBRAS

Let M be a von Neumann algebra acting in a Hilbert space $\mathcal{H}$. Let G be a locally compact topological group and let $\alpha : s \to \alpha_s$ be a homomorphism of G into the group of *-automorphisms of M such that for each $x \in M$ the map $s \to \alpha_s(x)$ is continuous from G into M where M is considered with its strong topology. Such a homomorphism is called a continuous action of G on M. In the case $G = \mathbf{R}$ a continuous action is of course a strongly continuous one-parameter group of *-automorphisms. Very often the triple (M, G, α) is called a covariant system. Remark that equivalently one can consider M with the weak, σ-weak, σ-strong or σ-strong* topology. A typical example, and in fact a very important one, is obtained in the case of a von Neumann algebra M acting in $\mathcal{H}$ and a continuous unitary representation $a : s \to a_s$ of G in $\mathcal{H}$ with the property that $a_s x a_s^* \in M$ for all $x \in M$ and $s \in G$. Then $\alpha_s(x) = a_s x a_s^*$ is easily seen to define a continuous action of G on M.

To a continuous action α of a locally compact group G on a von Neumann algebra M can be associated a new von Neumann algebra, called the crossed product of M by the action α of G, and denoted by $M \otimes_\alpha G$. Notations like $R(M, \alpha)$ and $W^*(M, G)$ are also used. In this section we will carefully introduce this new von Neumann algebra. We will also give some basic properties of the operators involved. First however we will need to study the Hilbert space in which it acts.

Let ds denote a left invariant Haar measure on G and let $L_2(G)$ be the Hilbert space of (equivalence classes of) square integrable functions from G into $\mathbf{C}$ with respect to the measure ds. Then the crossed product will be a von Neumann algebra acting in the Hilbert space tensor product $\mathcal{H} \otimes L_2(G)$ of $\mathcal{H}$ and $L_2(G)$. As we shall see, in the theory of crossed products, it is important to consider elements in $\mathcal{H} \otimes L_2(G)$ as $\mathcal{H}$-valued functions on G. Let us first make this precise.

2.1 **Notation.** Denote by $C_c(G, \mathcal{H})$ the complex vector space of $\mathcal{H}$-valued functions on G with compact support. Let the scalar product in $\mathcal{H}$ be denoted by $\langle \cdot, \cdot \rangle$. Then for every pair $\xi, \eta \in C_c(G, \mathcal{H})$, the function $s \to \langle \xi(s), \eta(s) \rangle$ will be a continuous, complex valued function with compact support in G. Then define

$$\langle \xi, \eta \rangle = \int \langle \xi(s), \eta(s) \rangle \, ds.$$

It is then easily verified that this expression gives a scalar product on $C_c(G, \mathcal{H})$. The completion of $C_c(G, \mathcal{H})$ with respect to this scalar product is denoted by $L_2(G, \mathcal{H})$.

It is justified to call this space $L_2(G, \mathcal{H})$ as it can be shown that the set of $\mathcal{H}$-valued functions ξ on G with the following properties:

(i) $\langle \xi(\cdot), \eta_0 \rangle$ is measurable for all $\eta_0 \in \mathcal{H}$,

(ii) there is a separable subspace $\mathcal{H}_0$ such that $\xi(s) \in \mathcal{H}_0$ for all $s \in G$,

(iii) $\| \xi(\cdot) \| \in L_2(G)$

is itself a Hilbert space with scalar product defined as above, and that $C_c(G, \mathcal{H})$ is a dense subspace of this Hilbert space.

However it turns out to be much more convenient to consider elements in $L_2(G, \mathcal{H})$ as elements of the completion of $C_c(G, \mathcal{H})$ than as functions. Therefore, as we will not need this result anyway, we refer to an appendix (A) for a proof of the above statement.

In fact, what is much more important in our treatment is that $L_2(G, \mathcal{H})$ can be canonically identified with $\mathcal{H} \otimes L_2(G)$. This is done in the following proposition.

2.2 **Proposition.** <u>There is an isomorphism</u> U <u>of</u> $\mathcal{H} \otimes L_2(G)$ <u>onto</u> $L_2(G, \mathcal{H})$ <u>such that</u>

$$(U(\xi_0 \otimes f))(s) = f(s)\xi_0$$

for any $\xi_0 \in \mathcal{H}$ and $f \in C_c(G)$, the set of complex-valued continuous functions with compact support in G.

Proof. Let $f_1, f_2, \dots, f_n \in C_c(G)$ and $\xi_1, \xi_2, \dots, \xi_n \in \mathcal{H}$. Define $\xi : G \to \mathcal{H}$ by $\xi(s) = \sum_{i=1}^{n} f_i(s)\xi_i$ for $s \in G$. Then clearly $\xi \in C_c(G, \mathcal{H})$ and

$$\begin{aligned}
\|\xi\|^2 &= \langle \xi, \xi \rangle = \int \langle \xi(s), \xi(s) \rangle ds \\
&= \sum_{i,j=1}^{n} \int f_i(s)\overline{f_j(s)} \langle \xi_i, \xi_j \rangle ds \\
&= \sum_{i,j=1}^{n} \langle \xi_i, \xi_j \rangle \langle f_i, f_j \rangle \\
&= \langle \sum_{i=1}^{n} \xi_i \otimes f_i, \sum_{j=1}^{n} \xi_j \otimes f_j \rangle \\
&= \| \sum_{i=1}^{n} \xi_i \otimes f_i \|^2 .
\end{aligned}$$

It follows that we can define a linear operator U from the algebraic tensor product of $\mathcal{H}$ with $C_c(G)$ into $C_c(G, \mathcal{H})$ by $U(\sum_{i=1}^{n} \xi_i \otimes f_i) = \xi$. Also U is isometric and therefore extends uniquely to an isometry from the Hilbert space tensor product $\mathcal{H} \otimes L_2(G)$ into $L_2(G, \mathcal{H})$, the extension is still denoted by U. It remains to show that U is onto, so that indeed U is an isomorphism. For this it is sufficient to show that functions of the form ξ above are dense in $C_c(G, \mathcal{H})$. So let $\xi_0 \in C_c(G, \mathcal{H})$, let K be the compact support of ξ_0 and let V be an open set with finite Haar measure such that $K \subseteq V$. Take $\varepsilon > 0$ and for each $s \in K$ choose a neighbourhood V_s of s such that $V_s \subseteq V$ and $\|\xi_0(t) - \xi_0(s)\| < \varepsilon$ for all $t \in V_s$. Then choose points $s_1, s_2 \dots s_n$ in K such that $K \subseteq V_{s_1} \cup V_{s_2} \dots \cup V_{s_n}$ and positive functions $h_1, h_2 \dots h_n$ in $C_c(G)$ such that the support of h_j lies in V_{s_j} for all $j = 1, n$, and such that

$$0 \leq \sum_{j=1}^{n} h_j(s) \leq 1 \quad \text{for all} \quad s \in G \quad \text{and} \quad \sum_{j=1}^{n} h_j(s) = 1$$

for all $s \in K$. Then if we define $\xi = \sum_{j=1}^{n} h_j \xi_0(s_j)$ we get

$$\begin{aligned} \|\xi(s) - \xi_0(s)\| &= \left\| \sum_{j=1}^{n} h_j(s)\xi_0(s_j) - \sum_{j=1}^{n} h_j(s)\xi_0(s) \right\| \\ &\leq \sum_{j=1}^{n} h_j(s) \|\xi_0(s_j) - \xi_0(s)\| \\ &\leq \sum_{j=1}^{n} h_j(s)\, \varepsilon \leq \varepsilon \end{aligned}$$

for all $s \in G$. Finally since ξ and ξ_0 have support in the set V we get

$$\|\xi - \xi_0\| \leq \varepsilon p^{\frac{1}{2}} \quad \text{where } p \text{ is the Haar measure of } V.$$

This proves the result.

In what follows we will always identify $L_2(G, \mathcal{H})$ and $\mathcal{H} \otimes L_2(G)$ by means of this isomorphism. So for any $\xi \in \mathcal{H}$ and $f \in C_c(G)$ we will consider $\xi \otimes f$ as a function on G with values in $\mathcal{H}$ given by $(\xi \otimes f)(s) = f(s)\xi$.

So far we have considered the space in which the crossed product is going to act. Now we come to the operators that will generate the crossed product.

2.3 Lemma. For every $x \in M$ and $\xi \in C_c(G, \mathcal{H})$ we have that the function ξ_1, defined by $\xi_1(s) = \alpha_{s^{-1}}(x)\xi(s)$ is again in $C_c(G, \mathcal{H})$. Moreover $\|\xi_1\| \leq \|x\| \|\xi\|$.

Proof. Remark first that indeed ξ_1 is a function of G in $\mathcal{H}$ as $\xi(s) \in \mathcal{H}$ for every s and $\alpha_{s^{-1}}(x) \in M$ and M acts in $\mathcal{H}$. Clearly ξ_1 has also compact support, so we must show continuity. This follows from the calculation below with s_0 fixed in G and $s \to s_0$.

Indeed
$$\begin{aligned} \|\xi_1(s) - \xi_1(s_0)\| &= \|\alpha_{s^{-1}}(x)\xi(s) - \alpha_{s_0^{-1}}(x)\xi(s_0)\| \\ &\leq \|\alpha_{s^{-1}}(x)(\xi(s) - \xi(s_0))\| + \|(\alpha_{s^{-1}}(x) - \alpha_{s_0^{-1}}(x))\xi(s_0)\| \\ &\leq \|x\| \|\xi(s) - \xi(s_0)\| + \|(\alpha_{s^{-1}}(x) - \alpha_{s_0^{-1}}(x))\xi(s_0)\| \end{aligned}$$

where the first term tends to zero as ξ is continuous and the second term because the action is continuous, that is $s \to \alpha_s(s)\xi_0$ is continuous, which is used here with $\xi_0 = \xi(s_0)$.

Finally

$$\begin{aligned} \|\xi_1\|^2 &= \int \|\xi_1(s)\|^2 ds \\ &= \int \|\alpha_{s^{-1}}(x)\xi(s)\|^2 ds \\ &\leq \int \|\alpha_{s^{-1}}(x)\|^2 \|\xi(s)\|^2 ds = \|x\|^2 \int \|\xi(s)\|^2 ds \\ &= \|x\|^2 \|\xi\|^2. \end{aligned}$$

Because of lemma 2.3 the following definition makes sense.

2.4 **Definition.** For every $x \in M$ we define a bounded operator $\pi(x)$ on $L_2(G, \mathcal{H})$ by

$$(\pi(x)\xi)(s) = \alpha_{s^{-1}}(x)\xi(s) \quad \text{for} \quad \xi \in C_c(G, \mathcal{H}).$$

If there are different actions around we will occasionally use π_α instead of π.

The crossed product will, among others, contain all the operators $\pi(x)$ with $x \in M$, therefore let us study them a little more.

2.5 **Proposition.** π <u>is a faithful normal</u> *-<u>representation of</u> M <u>in</u> $L_2(G, \mathcal{H})$.

Proof. Using the fact that α_s is a *-automorphism of M for each $s \in G$, a straightforward calculation shows that π is a *-representation.

Let us show that π is faithful, therefore assume $x \in M$ and $\pi(x) = 0$. Take $\xi_0 \in \mathcal{H}$ and $f \in C_c(G)$ and let $\xi = \xi_0 \otimes f$. Then

$$\begin{aligned} 0 = \langle \pi(x)\xi, \xi\rangle &= \int \langle (\pi(x)\xi)(s), \xi(s)\rangle ds = \int \langle \alpha_{s^{-1}}(x)\xi(s), \xi(s)\rangle ds \\ &= \int \langle \alpha_{s^{-1}}(x) f(s)\xi_0, f(s)\xi_0\rangle ds = \int \langle \alpha_{s^{-1}}(x)\xi_0, \xi_0\rangle |f(s)|^2 ds. \end{aligned}$$

Because this holds for all $f \in C_c(G)$, and because $\langle \alpha_{s^{-1}}(x)\xi_0, \xi_0\rangle$ is continuous in s, we have that $\langle \alpha_{s^{-1}}(x)\xi_0, \xi_0 \rangle = 0$ for all s, in particular $\langle x\xi_0, \xi_0 \rangle = 0$. Again this holds for all $\xi_0 \in \mathcal{H}$ so that $x = 0$. This proves that π is faithful.

To prove that π is normal, let $\{x_i\}_{i \in I}$ be a bounded, increasing net in M^+ with x as supremum. As π is a *-representation also $\pi(x_i)$ will be bounded and increasing and therefore will increase to some operator on $L_2(G, \mathcal{H})$, call it $\tilde{y}$. As $x_i \leq x$ for all i we also have $\pi(x_i) \leq \pi(x)$ so that $\tilde{y} \leq \pi(x)$. We must show $\tilde{y} = \pi(x)$.

First let $f \in C_c(G)$ and $\xi_0 \in \mathcal{H}$, then with $\xi = \xi_0 \otimes f$ we have as before

$$\langle \pi(x_i)\xi, \xi \rangle = \int \langle \alpha_{s^{-1}}(x_i)\xi_0, \xi_0 \rangle |f(s)|^2 ds.$$

Now $s \to \langle \alpha_{s^{-1}}(x_i)\xi_0, \xi_0 \rangle$ is a net of continuous positive functions, increasing to the function $s \to \langle \alpha_{s^{-1}}(x)\xi_0, \xi_0 \rangle$ which is also continuous. By Dini's theorem [10] we have uniform convergence on compacta. In particular we have uniform convergence on the support of f, so also the integrals will converge. Hence $\langle \pi(x_i)\xi, \xi \rangle \to \langle \pi(x)\xi, \xi \rangle$. So we have $\langle \pi(x)\xi, \xi \rangle = \langle \tilde{y}\xi, \xi \rangle$. Now as $\tilde{y} \leq \pi(x)$ it follows from $\langle (\pi(x) - \tilde{y})\xi, \xi \rangle = 0$ that also $(\pi(x) - \tilde{y})\xi = 0$ or $\pi(x)\xi = \tilde{y}\xi$. This implies $\pi(x) = \tilde{y}$ because functions of the form $\xi = \xi_0 \otimes f$ with $\xi_0 \in \mathcal{H}$ and $f \in C_c(G)$ span a dense subspace.

It is mainly because of the definition of $\pi(x)$ that we must work with $L_2(G, \mathcal{H})$ instead of $\mathcal{H} \otimes L_2(G)$. It is good to keep in mind that roughly speaking $\pi(x)$ is the multiplication operator in $L_2(G, \mathcal{H})$ by the function $s \to \alpha_{s^{-1}}(x)$. Multiplication operators will play an important role in the next section. Before we continue let us consider an example.

2.6 **Example.** Assume here that G is a finite group with n elements $\{s_1, s_2, \ldots, s_n\}$. Then using the appropriate normalization of the Haar measure, $L_2(G, \mathcal{H})$ can be identified with the direct sum $\mathcal{H} \oplus \mathcal{H} \oplus \ldots \mathcal{H}$ of n copies of $\mathcal{H}$ by means of the isomorphism

$$\xi \in L_2(G, \mathcal{H}) \to (\xi(s_1), \xi(s_2) \ldots \xi(s_n)).$$

Then in matrix notation we get the following expression for $\pi(x)$:

$$\pi(x) = \begin{pmatrix} \alpha_{s_1^{-1}}(x) & 0 & \cdots & 0 \\ 0 & \alpha_{s_2^{-1}}(x) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \alpha_{s_n^{-1}}(x) \end{pmatrix} .$$

In the next section we will obtain more information about the relation of $\pi(x)$ with the tensor product structure of $L_2(G, \mathcal{H})$. Now we define a representation of G in $L_2(G, \mathcal{H})$. This will provide us with the second type of operators that will generate the crossed product.

2.7 **Definition.** For every $t \in G$ we define a bounded operator $\lambda(t)$ on $L_2(G, \mathcal{H})$ by

$$(\lambda(t)\xi)(s) = \xi(t^{-1}s)$$

with $\xi \in C_c(G, \mathcal{H})$ and $s \in G$.

It follows easily from the invariance of the Haar measure that such an operator exists and is isometric.

2.8 **Proposition.** λ <u>is a strongly continuous unitary representation of</u> G <u>in</u> $L_2(G, \mathcal{H})$.

Proof. It is straightforward to verify that λ is indeed a representation of G and that $\lambda(t)$ is unitary for all $t \in G$. The proof of the strong continuity can either be obtained in a similar way as in the case $L_2(G)$, or can be obtained from the corresponding result in $L_2(G)$. Indeed, let $\xi = \xi_0 \otimes f$ with $\xi_0 \in \mathcal{H}$ and $f \in C_c(G)$. Then if we denote left translation by t^{-1} in $L_2(G)$ by λ_t we get

$$(\lambda(t)\xi)(s) = \xi(t^{-1}s) = f(t^{-1}s)\xi_0 = (\lambda_t f)(s)\xi_0 = (\xi_0 \otimes \lambda_t f)(s).$$

It follows that $\lambda(t) = 1 \otimes \lambda_t$ and the strong continuity in $L_2(G, \mathcal{H})$ follows from the one in $L_2(G)$ [11, p. 118].

The following relation shows that in the representation π the automorphisms α_t are unitarily implemented by the operators $\lambda(t)$.

2.9 **Lemma.** For all $x \in M$ and $t \in G$ we have $\lambda(t)\pi(x)\lambda(t)^* = \pi(\alpha_t(x))$.

Proof. Let $\xi \in C_c(G, \mathcal{H})$ and $s \in G$, then

$$(\lambda(t)\pi(x)\lambda(t)^*\xi)(s) = (\pi(x)\lambda(t)^*\xi)(t^{-1}s) = \alpha_{s^{-1}t}(x)(\lambda(t)^*\xi)(t^{-1}s)$$
$$= \alpha_{s^{-1}}(\alpha_t(x))\xi(s) = (\pi(\alpha_t(x))\xi)(s)$$

and the result follows.

It follows from this lemma that linear combinations of operators of the form $\pi(x)\lambda(t)$ with $x \in M$ and $t \in G$ form a *-algebra. Indeed, let $x, y \in M$ and $t, s \in G$ then

$$\begin{aligned}\pi(x)\lambda(t)\pi(y)\lambda(s) &= \pi(x)\lambda(t)\pi(y)\lambda(t)^*\lambda(t)\lambda(s)\\ &= \pi(x)\pi(\alpha_t(y))\lambda(ts)\\ &= \pi(x\alpha_t(y))\lambda(ts)\end{aligned}$$

and

$$(\pi(x)\lambda(t))^* = \lambda(t)^*\pi(x^*) = \lambda(t^{-1})\pi(x^*)\lambda(t^{-1})^*\lambda(t^{-1}) = \pi(\alpha_{t^{-1}}(x^*))\lambda(t^{-1}).$$

We now come to the definition of a crossed product.

2.10 **Definition.** The crossed product of M by the action α of G is the von Neumann algebra generated by the operators $\{\pi(x), \lambda(s) \mid x \in M, s \in G\}$ and is denoted by $M \otimes_\alpha G$. Because of the preceding it is the closure of the *-algebra of linear combinations of products $\pi(x)\lambda(s)$ with $x \in M$ and $s \in G$.

2.11 **Example.** Assume that G only has two elements $\{e, s\}$ where e is the identity. Then as in 2.6 the space $L_2(G, \mathcal{H})$ is identified with $\mathcal{H} \oplus \mathcal{H}$ and for $\pi(x)$ we get