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978-0-521-21543-5 - Pontryagin Duality and the Structure of Locally Compact Abelian Groups
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Excerpt
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1 · Introduction to topological groups

Definition. Let G be a set that is a group and a topological space. Then G is said to be a *topological group* if

(i) the mapping $(x,y) \to xy$ of $G \times G$ onto G is a continuous mapping of the cartesian product $G \times G$ (with the product topology) onto G;

and (ii) the mapping $x \to x^{-1}$ of G onto G is continuous.

Examples:

(1) The additive group of real numbers with the "usual" topology (i.e. that given by the metric $d(x,y) = |x-y|$). It will be denoted by R.

(2) The multiplicative group of positive real numbers with the "usual" topology.

(3) The additive group of rational numbers with the "usual" topology - denoted by Q.

(4) The group of integers with the discrete topology (i.e. every set is an open set) - denoted by Z.

(5) Any group with the discrete topology.

(6) Any group with the indiscrete topology (i.e. the open sets are ϕ and the whole space).

(7) The "circle group" consisting of the complex numbers of modulus one (i.e. the set of numbers $e^{2\pi ix}$, $0 \leq x < 1$) with the group operation being multiplication of complex numbers and topology induced from that of the complex plane. This topological group is denoted by T (or S^1).

(8) *Linear groups.* Let $A = (a_{jk})$ be an $n \times n$ matrix, where the coefficients a_{jk} are complex numbers. The

transpose ${}^tA$ of the matrix A is the matrix (a_{kj}) and the conjugate $\bar{A}$ of A is the matrix $(\bar{a}_{jk})$, where $\bar{a}_{jk}$ is the complex conjugate of the number a_{jk}. The matrix A is said to be *orthogonal* if $A = \bar{A}$ and ${}^tA = A^{-1}$ and *unitary* if $A^{-1} = {}^t(\bar{A})$.

The set of all non-singular $n \times n$ matrices (with complex number coefficients) is called the *general linear group* (over the complex number field) and is denoted by $GL(n,C)$. The subgroup of $GL(n,C)$ consisting of those matrices with determinant one is the *special linear group* (over the complex field) and is denoted by $SL(n,C)$. The *unitary group* $U(n)$ and the *orthogonal group* $O(n)$ consist of all unitary matrices and all orthogonal matrices, respectively; they are subgroups of $GL(n,C)$. Finally we define the *special unitary group* and the *special orthogonal group* as $SU(n) = SL(n,C) \cap U(n)$ and $SO(n) = SL(n,C) \cap O(n)$, respectively.

The group $GL(n,C)$ and all its subgroups can be regarded as subsets of C^{n^2} (where C denotes the complex number plane). As such $GL(n,C)$ and all its subgroups have induced topologies and it is easily verified that, with these, they are topological groups.

Remark. Of course not every topology on a group makes it into a topological group; i.e. the group structure and the topological structure need not be compatible.

Example. Let G be the group of integers. Define a topology on G as follows: a subset U of G is open if

(a) $0 \notin U$ or

(b) $G \setminus U$ is finite.

Clearly this is a (compact Hausdorff) topology but Proposition 1 will show that G is not a topological group.

Proposition 1. *Let G be a topological group. For each $a \in G$, left and right translation by a are homeomorphisms of G. Inversion is also a homeomorphism.*

Proof. The map $L_a: G \to G$ given by $g \to ag$ is the product of the two continuous maps

$G \to G \times G$ given by $g \to (a,g)$ where a is fixed

and

$G \times G \to G$ given by $(x,y) \to xy$

and is therefore continuous. So left translation by any $a \in G$ is continuous. Further, L_a has a continuous inverse, namely $L_{a^{-1}}$, since $L_a[L_{a^{-1}}(g)] = L_a[a^{-1}g] = a(a^{-1}g) = g$ and $L_{a^{-1}}[L_a(g)] = L_{a^{-1}}[ag] = a^{-1}(ag) = g$. So left translation is a homeomorphism. Similarly right translation is a homeomorphism.

The map $I: G \to G$ given by $g \to g^{-1}$ is continuous, by definition. Also I has a continuous inverse, namely I, itself, as $I[I(g)] = I[g^{-1}] = [g^{-1}]^{-1} = g$. So I is also a homeomorphism.//

It is now clear that our example above is not a topological group as left translation by 1 takes the open set $\{-1\}$ onto $\{0\}$, but $\{0\}$ is not an open set. What we are really saying is that any topological group is a homogeneous space while the example is not.

Definition. A topological space X is said to be *homogeneous* if it has the property that for each ordered pair x,y of points of X, there exists a homeomorphism $f: X \to X$ such that $f(x) = y$.

While every topological group is a homogeneous topological space, we will see shortly that not every homogeneous space can be made into a topological group.

Definition. A topological space is said to be a T_1-*space* if each point in the space is a closed set.

Definition. A topological space X is said to be *Hausdorff* or a T_2-*space* if for each pair of distinct points a and b in X, there exist open sets U_a and U_b, with $a \in U_a$, $b \in U_b$ and $U_a \cap U_b = \phi$.

It is readily seen that any Hausdorff space is a T_1-space but that the converse is false.

Example. Let X be any infinite set with the cofinite topology; that is, a subset U of X is open if and only if $U = X$, $U = \phi$ or $X \backslash U$ is finite.

Clearly this space is a T_1-space, but it is not Hausdorff as no (non-trivial) pair of open sets are disjoint.

We will see, however, that any topological group which is a T_1-space *is* Hausdorff. Incidentally, this is not true, in general, for homogeneous spaces - as the above example is homogeneous. As a consequence we will then have that not every homogeneous space can be made into a topological group.

Proposition 2. *Let G be any topological group and e its identity element. If U is any neighbourhood of e, then there exists an open neighbourhood V of e such that*

(i) $V = V^{-1}$ (that is, V is *symmetric*)

(ii) $V^2 \subseteq U$.

(Here $V^{-1} = \{v^{-1}: v \in V\}$ and

$V^2 = \{v_1v_2: v_1 \in V$ and $v_2 \in V\}$ (*not* the set $\{v^2: v \in V\}$).)

Proof. Exercise.

(Use the continuity of $x \to x^{-1}$ at $x = e$, and the continuity of $(x,y) \to xy$ at $(x,y) = (e,e)$.)//

Proposition 3. *Any topological group G which is a T_1-space is also a Hausdorff space.*

Proof. Let x and y be distinct points of G. Then $x^{-1}y \neq e$. The set $G \setminus \{x^{-1}y\}$ is an open neighbourhood of e and so, by Proposition 2, there exists an open symmetric neighbourhood V of e such that $V^2 \subseteq G \setminus \{x^{-1}y\}$. Thus $x^{-1}y \notin V^2$.

Now xV and yV are open neighbourhoods of x and y, respectively. Suppose $xV \cap yV \neq \phi$. Then $xv_1 = yv_2$, where v_1 and v_2 are in V; that is, $x^{-1}y = v_1 v_2{}^{-1} \in V.V^{-1} = V^2$ - which is a contradiction. Hence $xV \cap yV = \phi$ and so G is Hausdorff.//

So to check that a topological group is Hausdorff it is only necessary to verify that each point is a closed set. Indeed, by Proposition 1, it suffices to show that $\{e\}$ is a closed set.

Remark. Virtually all serious work on topological groups deals only with Hausdorff topological groups. (Indeed many authors include "Hausdorff" in their definition of topological group.) We will see one reason for this shortly. However, it is natural to ask: Does every group admit a Hausdorff topology which makes it into a topological group? The answer is obviously "yes" - the discrete topology. But we mention the following problem.

Question. Does every group admit a Hausdorff non-discrete group topology which makes it into a topological group?

S.Shelah (On a Kurosh problem: Jonsson groups; Frattini subgroups and untopologized groups) recently announced a negative answer, under the assumption of the continuum hypothesis. However in the special case that the group is abelian (= commutative) the answer is "yes" and to show this will be one of our earliest tasks.

EXERCISE SET ONE

1. Let G be a topological group, e its identity element, and k any element of G. If U is any neighbourhood of e, show that there exists an open neighbourhood V of e such that

(i) $V = V^{-1}$

and (ii) $V^2 \subseteq U$

and (iii) $k V k^{-1} \subseteq U$

(In fact, with more effort you can show that if K is a compact subset of G then V can be chosen to also have the property : (iv) for any $k \in K$, $k V k^{-1} \subseteq U$.)

2. (i) Let G be any group and let $\mathcal{N} = \{N\}$ be a family of normal subgroups of G. Show that the family of all sets of the form gN, as g runs through G and N runs through $\mathcal{N}$ is an open subbasis for a group topology on G. Such a topology is called a *subgroup topology*.

(ii) Prove that *every* group topology on a finite group is a subgroup topology with $\mathcal{N}$ consisting of precisely one normal subgroup.

3. A topological space X is said to be a T_0-*space* if given any x and y in X, either there exists an open set containing x but not y, or there exists an open set containing y but not x. A topological space X is said to be *regular* if for each $x \in X$ and each open neighbourhood U of x, there exists a closed neighbourhood V of x such that $V \subseteq U$. Show that

(i) any T_1-space is a T_0-space but that there exist T_0-spaces which are not T_1-spaces

(ii) every topological group is a regular space

(iii) any regular T_0-space is Hausdorff, and hence any topological group which is a T_0-space is Hausdorff.

4. Let G be a topological group, A and B subsets of G and g any element of G. Show that

(i) If A is open then gA is open.

(ii) If A is open and B is arbitrary, then AB is open.

(iii) If A and B are compact then AB is compact.

(iv) If A is compact and B is closed then AB is closed.

(v) If A and B are closed then AB need not be closed.

5. Let S be a compact subset of a metrizable topological group G, such that $xy \in S$ if x and y are in S. Show that for each $x \in S$, $xS = S$. (Let y be a cluster point of the sequence $x, x^2, x^3, \ldots$ in S and show that $yS = \bigcap_{n=1}^{\infty} x^n S$; deduce that $yxS = yS$.) Hence show that S is a subgroup of G. (Cf. Hewitt and Ross, *Abstract Harmonic Analysis* I, Theorem 9.16.)

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Definition. Let G_1 and G_2 be topological groups. A map $f: G_1 \to G_2$ is said to be a *continuous homomorphism* if it is both a homomorphism of groups and continuous. If f is also a homeomorphism then it is said to be a *topological group isomorphism* or a *topological isomorphism* and G_1 and G_2 are said to be *topologically isomorphic*.

Example. Let R be the additive group of real numbers with the usual topology and $R^{\times}$ the multiplicative group of positive real numbers with the usual topology. Then R and $R^{\times}$ are topologically isomorphic, where the topological isomorphism $R \to R^{\times}$ is $x \to \exp(x)$. (Hence we need not mention this group $R^{\times}$ again since, as topological groups, R and $R^{\times}$ are the same.)

Proposition 4. *Let G be a topological group and H a subgroup of G. With its relative topology as a subset of G, H is a topological group.*

Proof. The mapping $(x,y) \to xy$ of $H \times H$ onto H and the mapping $x \to x^{-1}$ of H onto H are continuous since they are restrictions of the corresponding mappings of $G \times G$ and G . //

Examples.

(i) $Z \leq R$.

(ii) $Q \leq R$.

Proposition 5. *Let H be a subgroup of a topological group G. Then*

(i) *the closure $\bar{H}$ of H is a subgroup of G ;*

(ii) *if H is a normal subgroup of G then $\bar{H}$ is a normal subgroup of G ;*

(iii) *if G is Hausdorff and H is abelian, then $\bar{H}$ is abelian.*

Proof. Exercise.

Corollary. *Let G be a topological group. Then $\overline{\{e\}}$ is a closed normal subgroup of G ; indeed, it is the smallest closed subgroup of G. If $g \in G$, then $\overline{\{g\}}$ is the coset $g\overline{\{e\}} = \overline{\{e\}}g$. (Of course if G is Hausdorff then $\overline{\{e\}} = \{e\}$.)*

Proof. This follows immediately from Proposition 5 (ii) by noting that $\{e\}$ is a normal subgroup of G . //

Proposition 6. *Any open subgroup* H *of a topological group* G *is (also) closed.*

Proof. Let x_i , $i \in I$ be a set of right coset representatives of H in G . So $G = \bigcup_{i\in I} Hx_i$, where $Hx_i \cap Hx_j = \phi$, for any distinct i and j in the index set I . Since H is open, so is Hx_i open, for each $i \in I$. Of course for some $i_0 \in I$, $Hx_{i_0} = H$, that is, we have

$$G = H \cup \left(\bigcup_{i\in J} Hx_i \right) , \quad \text{where } J = I \setminus \{i_0\} .$$

These two terms are disjoint and the second term, being the union of open sets, is open. So H is the complement (in G) of an open set, and is therefore closed in G . //

Note that the converse of Proposition 6 is false. For example, Z is a closed subgroup of R , but it is not an open subgroup of R .

Proposition 7. *Let* H *be a subgroup of a Hausdorff group* G *. If* H *is locally compact, then* H *is closed in* G *. In particular this is the case if* H *is discrete.*

Proof. Let K be a compact neighbourhood in H of e . Then there exists a neighbourhood U in G of e such that $U \cap H = K$. In particular, $U \cap H$ is closed in G . Let V be a neighbourhood in G of e such that $V^2 \subseteq U$.

If $x \in \bar{H}$, then as $\bar{H}$ is a group (Proposition 5), $x^{-1} \in \bar{H}$. So there exists an element $y \in Vx^{-1} \cap H$. We will show that $yx \in H$. As $y \in H$, this will imply that $x \in H$ and hence H is closed, as required.

To show that $yx \in H$ we verify that yx is a limit point of $U \cap H$. As $U \cap H$ is closed this will imply that $yx \in U \cap H$ and so, in particular, $yx \in H$.

Let O be an arbitrary neighbourhood of yx. Then $y^{-1}O$ is a neighbourhood of x, and so $y^{-1}O \cap xV$ is a neighbourhood of x. As $x \in \bar{H}$, there is an element $h \in (y^{-1}O \cap xV) \cap H$. So $yh \in O$. Also $yh \in (Vx^{-1})(xV) = V^2 \subseteq U$, and $yh \in H$; that is, $yh \in O \cap (U \cap H)$. As O is arbitrary, this says that yx is a limit point of $U \cap H$, as required. //

Proposition 8. *Let* U *be a symmetric neighbourhood of* e *in a topological group* G. *Then* $H = \bigcup_{n=1}^{\infty} U^n$ *is an open (and closed) subgroup of* G.

Proof. Clearly H is a subgroup of G. Let $h \in H$. Then $h \in U^n$, for some n. So $h \in hU \subseteq U^{n+1} \subseteq H$; that is, H contains the neighbourhood hU of h. As h was an arbitrary element of H, H is open in G. It is also closed in G, by Proposition 6. //

Corollary 1. *Let* U *be any neighbourhood of* e *in a connected topological group* G. *Then* $G = \bigcup_{n=1}^{\infty} U^n$; *that is, any connected group is generated by any neighbourhood of* e.

Proof. Let V be a symmetric neighbourhood of e such that $V \subseteq U$. By Proposition 8, $H = \bigcup_{n=1}^{\infty} V^n$ is an open and closed subgroup of G.

As G is connected, $H = G$; that is $G = \bigcup_{n=1}^{\infty} V^n$. As $V \subseteq U$, $V^n \subseteq U^n$, for each n and so $G = \bigcup_{n=1}^{\infty} U^n$, as required. //

Definition. A topological group G is said to be *compactly generated* if there exists a compact subset X of G such that G is the smallest subgroup (of G) containing X.