Preface

These notes address the connection between two subjects, and they are thus intended to form an introduction to both but to be about neither. The discoveries of Fefferman and Stein about H^p and BMO have interacted fruitfully with a great deal of work on the analogous ideas in martingale theory; the main goal of the following pages is an explanation of the fundamental result of Burkholder, Gundy, and Silverstein, which forms the bridge between these two areas of investigation. The exposition is at as elementary a level as possible, and it is intended in particular to be accessible to graduate students with a basic knowledge of measure theory, complex analysis and functional analysis. For the sake of those not familiar with probability theory, many probabilistic results are introduced and proved as needed, and there is a chapter without proofs on Brownian motion. Again, for those not on everyday terms with classical function theory, a survey of results on the maximal, square, and Littlewood-Paley functions is included, and function-theoretic arguments are given and estimates made in considerable detail. The discussion is restricted mainly to the case of the unit disk in the complex plane. I hope that one who reads these notes will find that Garsia's book, the papers of Fefferman and Stein, and the writings of Burkholder, Davis, Gundy, Herz, Silverstein, et al. on these topics are easily approachable.

I originally organized this material for a series of seminar talks that I gave at U. N. C. in the fall of 1975, and it is a pleasure to thank the participants for all that they taught me. The review of the basic properties of Brownian motion is based in part on lectures given by S. Kakutani, and I am grateful to R. F. Gundy for an outline of much of the material in Chapter 7. I would also like to thank Janet Farrell for her quick and accurate typing of the original manuscript and the National Science Foundation for support during the period that these notes were in preparation.

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1·Introduction

In 1915 G. H. Hardy, answering a question of Bohr and Landau, investigated properties of the mean over a circle of the modulus $|F|$ of an analytic function F which were similar to those of the maximum value of $|F|$ over a disk. He found that his results applied also to $|F|^p$ for $p > 0$, and thus was founded the theory of H^p spaces. Since then these Hardy spaces have been the object of much research, and their connections with such diverse subjects as classical function theory (especially the boundary behavior of analytic functions), potential theory (including the theory of harmonic functions and partial differential equations), Fourier series, functional analysis, and operator theory (for example Beurling's work on invariant subspaces of the shift operator) have been developed in considerable detail.

An entirely new line of investigation for the Hardy spaces was uncovered in 1971 by Burkholder, Gundy, and Silverstein when they showed that for $0 < p < \infty$ an analytic function $F = u + i\tilde{u}$ is in H^p if and only if the maximal function of u is in L^p. Surprisingly, their arguments were probabilistic in nature, being carried out by manipulation of Brownian motion in the complex plane. Their result showed that the Hardy spaces could be characterized in real-variable terms and thus H^p theory could be easily extended to higher dimensions and more general kinds of spaces.

Such a generalization and extension were accomplished in 1972 by Fefferman and Stein with analytic rather than probabilistic arguments, in which a careful study of the Lusin S-function (which is also called the 'square function' and 'area function') played an important part. (Their work built on the efforts of many authors in addition to Burkholder, Gundy, and Silverstein, including, for example, Calderón, Segovia, Stein and Weiss, and Zygmund; it would take us too far afield, however, to attempt to indicate in a historically correct manner the contributions of the many researchers who have dealt with these problems.) They were able to extend the Burkholder-Gundy-Silverstein result to higher dimensions, give a real-variable characterization of H^p, and identify the dual space of H^1 as the space BMO of all functions of bounded mean oscillation. In particular, they showed that for any analytic function $F = u + i\tilde{u}$ and any p with $0 < p < \infty$, the following seven statements are equivalent: (1) $F \in H^p$ (for $p \geq 1$ this is equivalent to the existence of an L^p boundary function); (2) the classical maximal function $N_\sigma F$ (whose values are the least upper bounds of $|F|$ over Stolz domains) is in L^p; (3) the Lusin S-function $S(F)$ of F is in L^p; (4) $S(u) \in L^p$ (trivial, since $S(u) = S(F)$); (5) $N_\sigma u \in L^p$; (6) (for $p \geq 1$) if ϕ is the boundary function of u and $\mathcal{P}_r$ is the Poisson kernel, then $\sup_{0 \leq r < 1} |\mathcal{P}_r * \phi(x)| \in L^p$; and (7) (for $p \geq 1$) $\sup_{0 \leq r < 1} |\psi_r * \phi(x)| \in L^p$ for each reasonably smooth approximate identity $\{\psi_r\}$.

The duality between H^1 and BMO makes possible other real-variable characterizations of H^p. Denote by Γ the unit circle in the complex plane and by m normalized Lebesgue measure on Γ. A real-valued function $a \in L^\infty(\Gamma)$ is called an <u>atom</u> in case

(i) a is supported on an interval I and $\|a\|_\infty \leq \frac{1}{m(I)}$,

(ii) a takes no more than two non-zero values, and

(iii) $\int a\,dm = 0$.

Let u be a real-valued function in $L^1(\Gamma)$. Then u is the real part of the boundary function of some $f \in H^1$ if and only if there are a sequence $a_1, a_2, \ldots$ of atoms and a sequence $\lambda_1, \lambda_2, \ldots$ of real numbers such that $\sum |\lambda_n| < \infty$ and $u = \int u\,dm + \sum \lambda_n a_n$. Moreover, letting $\lambda(u)$ equal the infimum of $\sum |\lambda_n|$ over all such sequences $\lambda_1, \lambda_2, \ldots$, there are universal constants c_1 and c_2 such that $c_1 \|f\|_1 \leq |\int u\,dm| + \lambda(u) \leq c_2 \|f\|_1$. This striking result is also due to Fefferman. A very simple proof has been given by Axler [2], and Coifman [10] has published a proof for the case $0 < p \leq 1$.

Probabilistic proofs are possible in this area of analysis because of the connection between probability theory and potential theory, which is seen in Kakutani's theorem equating harmonic measure with the hitting probabilities of Brownian motion. For a point $z \in D$ in the unit disk D

of the complex plane and a measurable subset $A \subset \Gamma$ of the boundary Γ of D, the harmonic measure $\omega_z(A)$ of A at z is defined to be the value at z of the harmonic function in D whose boundary values are 1 on A and 0 on $\Gamma \backslash A$; thus

$$\omega_z(A) = \frac{1}{2\pi} \int_A \mathcal{P}(r, \theta - t)dt,$$

where $z = re^{i\theta}$ and $\mathcal{P}(r, \theta - t) = \dfrac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2}$ is the Poisson kernel. Kakutani proved in 1944 that $\omega_z(A)$ equals the probability that a Brownian traveler starting from the point z, at the time that he first hits Γ, hits Γ in a point of A. Even more, if S is any bounded, connected, open subset of $\mathbf{R}^n$ with smooth boundary ∂S, f is continuous on ∂S, $\gamma_{z,t}$ is a Brownian motion starting at $z \in S$, and $T(z) = \inf\{t \geq 0 : \gamma_{z,t} \notin S\}$, then $u(z) = E(f(\gamma_{z, T(z)}))$ is harmonic in S and $\lim_{z \to x} u(z) = f(x)$ for each $x \in S$ (that is, u is the solution in S of the Dirichlet problem with boundary values f). Besides its applications to analysis, this connection also led to the development of probabilistic potential theory by Doob, Hunt, and others.

Similarly, many of the classical arguments involving the maximal and square functions have been carried forward also in a probabilistic setting, and now even the inequalities involving H^p and BMO norms are seen to apply also to abstract martingales.

The main purpose of these notes is an investigation of the relationship of these new martingale inequalities to the analytic results which they mirror, and therefore the major portion is taken up by an exposition of the theorem of Burkholder, Gundy, and Silverstein. It will be seen that in many of the proofs translations are made from the probabilistic to the analytic setting by changes of variables in accordance with Kakutani's theorem. We begin with a historical survey of results related to the Hardy-Littlewood maximal function, the Lusin S-function, and the Littlewood-Paley g-function, and Chapter 3 gives a short list of relevant definitions and theorems related to Brownian motion. The concluding chapter justifies the probabilistic definitions of H^p and BMO and includes a proof of the H^1-BMO duality theorem via continuous-time martingales. I hope that the reader will go then to the original literature for the full details of this developing story.

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2 · The maximal, square and Littlewood-Paley functions

D denotes the unit disk $\{z \in \mathbf{C} : |z| < 1\}$ in the complex plane, $\Gamma = \{e^{i\theta} : 0 \le \theta < 2\pi\}$ its boundary, and m normalized Lebesgue measure on Γ. For a function f defined on D, $0 < p < \infty$, and $0 \le r < 1$, we define

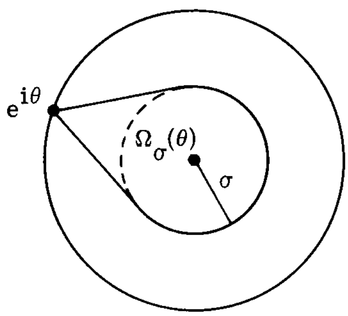
$$M_p(r, f) = [\int_0^{2\pi} |f(re^{i\theta})|^p dm(\theta)]^{1/p} .$$

An analytic function f on D is said to be of the class H^p in case

$$\sup_{r<1} M_p(r, f) = \|f\|_p < \infty .$$

The space of bounded analytic functions on D, with the supremum norm, is denoted by H^∞. Each (real-valued) harmonic function u on D has a unique <u>conjugate function</u> $\tilde{u}$ determined by the conditions that $\tilde{u}(0) = 0$ and $u + i\tilde{u}$ is analytic on D.

For a fixed σ with $0 < \sigma < 1$, each point $e^{i\theta} \in \Gamma$ determines a <u>Stolz domain</u> $\Omega_\sigma(\theta)$, as shown in the figure:



$\Omega_\sigma(\theta)$ is the interior of the convex hull of the circle of radius σ with center at the origin and the point $e^{i\theta}$.

Given any function f defined on D, the (classical) maximal function $N_\sigma f$ of f is defined by

$$N_\sigma f(e^{i\theta}) = \sup\{\,|f(z)| : z \in \Omega_\sigma(\theta)\,\}.$$

The square function $S_\sigma F$ of an analytic function F on D is

$$S_\sigma F(e^{i\theta}) = [\int\int_{\Omega_\sigma(\theta)} |F'(z)|^2 dxdy]^{\frac{1}{2}}.$$

The square of $S_\sigma F(e^{i\theta})$ equals the area of the image of $\Omega_\sigma(\theta)$ under F. The Littlewood-Paley function g_F of an analytic function F on D is defined by

$$g_F(e^{i\theta}) = [\int_0^1 (1-r)|F'(re^{i\theta})|^2 dr]^{\frac{1}{2}}.$$

Each of these functions measures in its own way the growth of the given function near the boundary of D. It has been proved that for an analytic function F, and for almost all $e^{i\theta} \in \Gamma$, the existence of a nontangential limit for F at $e^{i\theta}$ is equivalent to the finiteness of $N_\sigma F(e^{i\theta})$, or alternatively, to that of $S_\sigma F(e^{i\theta})$. Also, many inequalities among the norms of F, $N_\sigma F$, $S_\sigma F$, and g_F have been established, and these results have been carried over to the theory of martingales. Some idea of the history of the subject may be obtained from the following list of several of the important discoveries concerning these functions. The obvious conventions with regard to inequalities are in force when one side happens to be infinite.

2.1 G. H. Hardy and J. E. Littlewood, 1930 [24]

For each p with $0 < p < \infty$ there is a constant c_p such that whenever $F = u + i\tilde{u}$ is analytic on D,

$$\int_0^{2\pi} |N_\sigma F(e^{i\theta})|^p dm(\theta) \le c_p \sup_{0\le r<1} \int_0^{2\pi} |F(re^{i\theta})|^p dm(\theta),$$

or in more compact notation, $\|N_\sigma F\|_{L^p(\Gamma)} \le c_p \|F\|_{H^p}$. Hence also $\|N_\sigma u\|_{L^p(\Gamma)} \le c_p \|F\|_{H^p}$, and so this result comprises half of the Burkholder-Gundy-Silverstein Theorem, whose probabilistic proof we will examine below. (The Hardy-Littlewood proof relied on a combinatorial lemma and imagery referring to the game of cricket.)

In the same paper Hardy and Littlewood also considered a maximal function $M\phi$ defined for nonnegative $\phi \in L^1(\Gamma)$ by

$$M\phi(e^{i\theta}) = \sup_{e^{i\theta} \in I} \frac{1}{m(I)} \int_I \phi(e^{i\theta})dm(\theta)$$

and proved that for each $p > 1$ there is a constant c_p such that

$$\|M\phi\|_p \le c_p \|\phi\|_p .$$

2.2 **N. Lusin,** 1930 [41]

For each σ with $0 < \sigma < 1$ there is a constant A_σ such that if $F(z) = \sum_{n=0}^{\infty} a_n z^n$ is in H^2, then

$$\int_0^{2\pi} [S_\sigma F(e^{i\theta})]^2 dm(\theta) \le A_\sigma \sum_{n=0}^{\infty} |a_n|^2 .$$

It follows, then, that $S_\sigma F(e^{i\theta}) < \infty$ for almost all θ.

2.3 **J. Marcinkiewicz and A. Zygmund,** 1938 [42]

(i) For each σ with $0 < \sigma < 1$ and p with $0 < p < \infty$ there is a constant $A_{p,\sigma}$ such that

$$\|S_\sigma F\|_{L^p(\Gamma)} \le A_{p,\sigma} \|F\|_{H^p}$$

for each analytic function F on D.

(ii) If F is analytic on D and has a nontangential limit at every point of a subset E of Γ which has positive measure, then $S_\sigma F(e^{i\theta}) < \infty$ a.e. on E.

D. Spencer, 1943 [52]

Let F be analytic on D. If $E \subset \Gamma$ has positive measure and for each $e^{i\theta} \in E$ there is $\sigma = \sigma(\theta)$ with $0 < \sigma < 1$ such that $S_\sigma F(e^{i\theta}) < \infty$, then F has a nontangential limit at almost every point of E.

2.4 **J. E. Littlewood and R. E. A. C. Paley,** 1931, 1936, 1937 [37, 38, 39]

For each p with $0 < p \leq \infty$ there is a constant C_p such that if F is analytic on D then

$$\|g_F\|_{L^p(\Gamma)} \leq C_p \|F\|_{H^p}.$$

Conversely, for $1 < p < \infty$ there is a constant B_p such that for each analytic F on D

$$\|F\|_{H^p} \leq B_p \|g_F\|_{L^p(\Gamma)} .$$

J. Marcinkiewicz and A. Zygmund, 1938 [42]

For each σ with $0 < \sigma < 1$ there is a constant A_σ such that if F is analytic on D then

$$g_F(e^{i\theta}) \leq A_\sigma S_\sigma F(e^{i\theta}) \text{ for all } \theta \in [0, 2\pi).$$

G. Gasper, Jr., 1968 [22]

extended some of these results on g_F and $S_\sigma F$ to higher dimensions.

2.5 **M. Riesz,** 1920 [49]

For $0 < p < \infty$ let h^p denote the space of all real-valued harmonic functions u in D for which

$$\|u\|_{h^p} = \sup_{0 \leq r < 1} M_p(r, u) < \infty ,$$

and let h^∞ denote the space of all bounded real-valued harmonic functions on D with the supremum norm.

Then for each p with $1 < p < \infty$ there is a constant C_p such that

$$\|\tilde{u}\|_{h^p} \leq C_p \|u\|_{h^p} \text{ for all } u \in h^p.$$