PART ONE

GENERATORS AND RELATIONS FOR GROUPS OF HOMEOMORPHISMS

HERBERT ABELS

The aim of the present paper is to unify and generalize the proofs of results of Behr, Gerstenhaber and Macbeath concerning the theme of the title.

I. RESULTS

1.1 NOTATIONS. Let the group G act on the topological space X, i.e. suppose a homeomorphism of G into the group of homeomorphisms of X is given. By gx we denote the image of the point $x \in X$ under the homeomorphism corresponding to $g \in G$. For $M \subset G$, $A \subset X$ let $MA = \{gx;\ g \in M,\ x \in A\}$.

For any two subsets A, B of X define

(1.1.1) $$G(A,B) := \{g \in G;\ gA \cap B \neq \emptyset\}.$$

Obviously

(1.1.2) $$G(B,A) = [G(A,B)]^{-1}$$

(1.1.3) $$G(A,gB) = g\cdot G(A,B)$$

(1.1.4) $$G(gA,B) = G(A,B)\cdot g^{-1}$$

Let F be a non empty subset of X (think of F as a "fundamental set"). Define

(1.1.5) $$E := G(F,F)$$

(think of E "Erzeugendenmenge"). Let H be a group, $q:H \to G$ be a homomorphism. Suppose a section $s:E \to H$ is given, i.e. a map $s:E \to H$

such that $q \circ s = \mathrm{id}|_E$.

In applications H will always be a group with generators E and certain relations, which hold in G, $q:H \to G$ will be the homomorphism induced by the inclusion $E \to G$, the section s is the obvious one. The problem is: Under which conditions is q an isomorphism? We express the relations that hold in H by multiplicative properties of the section s.

1.2 HYPOTHESES

(1.2.1) (<u>Multiplicative hypothesis</u>) If $g_1F \cap g_2F \cap F \neq \emptyset$ we have

$$s(g_1^{-1})\ s(g_2) = s(g_1^{-1}g_2).$$

Both sides of this equality are defined because $E = E^{-1}$ by 1.1.2 and $g_1^{-1}g_2 \in G(F,F) = E$.

Further hypotheses are:

(1.2.2) $GF = X$

(1.2.3) $s(E)$ generates H

(1.2.4) F is connected

(1.2.5) X is connected and simply connected.

1.3 RESULTS

THEOREM 1. $q:H \to G$ <u>is an isomorphism if</u> (1.2.1) <u>through</u> (1.2.5) <u>hold and</u> $G\overset{\circ}{F} = X$ (<u>e.g. if F is open</u>).

THEOREM 2. $q:H \to G$ <u>is an isomorphism if</u> (1.2.1) <u>through</u> (1.2.5) <u>hold</u>, F <u>is closed in</u> X <u>and</u> $\{gF;\ g \in G\}$ <u>is a locally finite cover of</u> X.

Macbeath [6] proved Theorem 1 for open F, Theorem 2 for groups of isometries. The case of finite E in Theorem 2 was proved by Behr [1], with a bigger set of relations. Cf also [5]. Swan

[11] considers the case F open, $\pi_0(F) = \pi_0(X) = 0$ and gives a detailed description of ker(q) if $\pi_1(X) \neq 0$.

We actually prove a common generalization of Theorems 1 and 2 (Theorem 4.5). A result of Soulé [10] is an easy application (see 4.6).

For our next result we need to following definition. The cover $\{gF;\ g \in G\}$ of X is called <u>G-numerable</u> if there is a partition of unity $\{p_g;\ g \in G\}$ with $\mathrm{supp}(p_g) \subset gF$ such that $p_g(gx) = p_e(x)$ for every $x \in X$, $g \in G$. For example, if X is normal, F is open and $\{gF;\ g \in G\}$ is a locally finite cover of X, or if X is normal and F is a neighbourhood of a closed subset of F' and $\{gF';\ g \in G\}$ is a locally finite cover of X, then $\{gF;\ g \in G\}$ is a G-numerable cover of X.

THEOREM 3. $q: H \to G$ <u>is an isomorphism if</u> (1.2.1) <u>and</u> (1.2.3) <u>hold</u>, $\{gF;\ g \in G\}$ <u>is a G-numerable cover of</u> X <u>and</u> $\pi_0(F) = \pi_0(X) = \pi_1(X) = 0$.

1.4 IDEA OF PROOF. It is easy to prove that q is surjective (see Section II). If in Theorems 1 and 2 we drop the assumption that X be simply connected, we obtain a covering space $Y \xrightarrow{p} X$ (i.e. a locally trivial sheaf) with the following properties:

(1) H acts on Y and p is an H-map, i.e. $p(hy) = q(h)p(y)$.

(2) ker q acts as a group of covering transformations of p. The action is free and transitive on the fibres of p.

(3) There is a section for p over F.

Hence if any such covering space of X is trivial, $q: H \to G$ is an isomorphism, in particular if X is simply connected.

The main difficulty of the proof is to define a topology on Y.

The proof of Theorem 3 makes use of the nerve of the covering and its geometric realization. It makes use of fundamental groups instead of covering spaces. There is a similar generalization as above (see Theorem 5.4).

I thank H. Behr for helpful conversations. This paper actually grew out of a talk in a seminar of Behr's. I also thank the referee for useful hints to the literature.

II SURJECTIVITY OF q

Notations as in 1.1. The following result is well known [9, no.9].

2.1. THEOREM. *Suppose* $GF = X$, EF *is a neighbourhood of* F *and* X *is connected*. *Then* E *generates* G.

PROOF. Let G_o be the subgroup of G generated by E. The set $\{X_i = g_iG_oF;\ g_iG_o \in G/G_o\}$ is a cover of X, since $GF = X$. The X_i's are disjoint: $g_1G_oF \cap g_2G_oF \neq \emptyset$ implies that $G_og_2^{-1}g_1G_o$ contains an element of $E \subset G_o$, so $g_2^{-1}g_1 \in G$, hence $g_1G_o = g_2G_o$. Since EF is a neighbourhood of F, each $X_i = g_iG_oF = g_iG_o(EF)$ is a neighbourhood of itself, i.e. open. So the $X_i = g_iG_oF$ form an open disjoint cover of X. If X is connected, there is only one of them: $\# G/G_o = 1$, so $G = G_o$.

III THE SET Y

For the whole Section III we use the notations of 1.1 and assume only the multiplicative hypothesis (1.2.1): If $g_1F \cap g_2F \cap F \neq \emptyset$ we have

(3.1) $$s(g_1^{-1})\, s(g_2) = s(g_1^{-1}\, g_2).$$

This implies for $g_1 = g_2 = e$ the neutral element

(3.2) $$s(e) = e.$$

For $g_1\, g_2 = e$ we obtain

(3,3) $$s(g^{-1}) = (s(g))^{-1} \quad \text{for } g \in E,$$

which we sometimes denote by $s(g)^{-1}$.

We have an action of H on X defined by

(3.4) $$hx := q(h)x, \quad h \in H,\ x \in X.$$

Define

(3.5) $$Z := \{(x,h) \in X \times H; h^{-1}x \in F\}.$$

The relation on Z

(3.6) $(x_1,h_1) \sim (x_2,h_2)$ if and only if $x_1 = x_2$ and $h_1^{-1}\, h_2 \in s(E)$

is an equivalence relation by our multiplicative hypotheis (3.1). We define

(3.7) $$Y = Z/\sim.$$

The main point of the proof will be to endow Y with a suitable topology. We need some preparations. We denote the equivalence class of $(x,h) \in Z$ by $[x,h]$. We have an action of H on Y defined by $h_1[x,h] = [h_1x, h_1\, h]$. The projection $p:Y \to X$, $p([x,h]) = x$ is an H-map. We have a section $t:F \to Y$, namely

(3.8) $$t(x) = [x,e] \quad \text{for } x \in F.$$

Our definitions imply

(3.9) $$H\,t(F) = Y.$$

(3.10) $$H(t(x), t(F)) = s(G(x,F)) \text{ for } x \in F,$$

hence

(3.11) $$H(t(F),\ t(F)) = s(E).$$

The homomorphism $q:H \to G$ is to be analysed. Set $K = \ker q$. The next lemma shows that K is a good candidate for the group of covering transformations of p.

3.12 LEMMA. K acts freely on Y and simply transitively on the non-empty fibres $p^{-1}(x)$ of p.

PROOF. We have to show first that $K_y = \{k \in K;\ ky = y\}$ contains only the neutral element. By (3.9) it suffices to prove that claim for $y \in t(F)$, say $y = [x,e]$. If $ky = [kx,k] = [x,e]$, we have $k \in s(E) \cap K$. But $s(E) \cap K = \{e\}$, since $q|\,s(E)\colon s(E) \to E$ is bijective with inverse mapping $g \to s(g)$.

K acts on the fibres of p. It remains to prove that K acts transitively on the non empty fibres $p^{-1}p(y)$ of p, $y \in Y$. Again, we may assume, $y = [x,e]$. Suppose $z \in p^{-1}p(y)$. By (3.9) there is an element $h \in H$ and a point $[x_1,e] \in t(F)$ such that $z = h\,[x_1,e] = [hx_1,g]$. Since $p(y) = p(z)$ we have $x = hx_1 = q(h)x_1$, so $q(h) \in E$. For $h_1 = s(q(h)) \in s(E)$ we have $h_1[x_1,e] = [x,e] = y$. Hence $z = h[x_1,e] = h \cdot h_1^{-1}$ and $k = h \cdot h_1^{-1} \in K$.

Note that k is the unique element of K such that $ky = z$. In particular: $hs(q(h))^{-1}$ is the same element of K for every $h \in H$ such that $z = [x,h]$. So the proof actually yields the inverse mapping of

CAMBRIDGE

Cambridge University Press
978-0-521-21509-1 - Transformation Groups: Proceedings of the Conference in the University of Newcastle upon Tyne, August 1976
Edited By Czes Kosniowski
Excerpt
More information

$$\begin{array}{llcl} & K \times F & \to & p^{-1}(F) \\ (3.13) & (k, x) & \to & kt(x) = [x, k] \\ \text{namely} & (h\ s(q(h))^{-1},\ x) & \leftarrow & [x, h]. \end{array}$$

The next lemma makes explicit the properties of the topology of Y we want.

3.14 LEMMA. *Suppose* Y *is endowed with a topology such that*

(a) *Every* $h \in H$ *acts as a homeomorphism on* Y.

(b) $p:Y \to X$ *is a sheaf*. (*i.e. a local homeomorphism, i.e. every point* $y \in Y$ *has an open neighbourhood* U *such that* $p|U : U \to p(U)$ *is a homeomorphism and* $p(U)$ *is open in* X_1).

Then $p:Y \to X$ *is a covering, i.e. a locally trivial sheaf*, K *acts as a group of covering transformations of* p, *transitively on the non empty fibres of* p.

PROOF. Suppose U as in (b). Then $p^{-1}(p(U)) = \bigcup_{k \in K} kU$ is the disjoint union of the open sets kU. Endow K with the discrete topology. In the commutative diagram

$$\begin{tikzcd} & K \times U \arrow[dl] \arrow[dr] & \\ K \times p(U) \arrow[rr] \arrow[dr] & & p^{-1}(p(U)) \arrow[dl] \\ & p(U) & \end{tikzcd}$$

the upper diagonal maps are homeomorphisms, hence so is the horizontal map, yielding the local triviality of the sheaf $p:Y \to X$.

We need a more technical version of 3.14. We call a map $r:A \to Y$ a *section* (for p) if $p \circ r = id_A$. Note that Y is not supposed to have a topology yet. If $r:A \to Y$ is a section, so is ${}_h r:hA \to Y$

Cambridge University Press
978-0-521-21509-1 - Transformation Groups: Proceedings of the Conference in the University of Newcastle upon Tyne, August 1976
Edited By Czes Kosniowski
Excerpt
More information

defined by

(3.15) $$_h r(hx) = hr(x).$$

As usual, two maps defined in neighbourhoods of the same point $x \in X$ are said to have the same germ at $x \in X$, if and only if they coincide in some neighbourhood of x.

3.16 LEMMA. *Suppose we are given for every point* $x \in X$ *an open neighbourhood* U_x *of* x *and a section* $r_x : E_x \to Y$ *with the following properties:*

(a) $$r_x | U_x \cap F = t | U_x \cap F$$

(b) *If* $x_1 \in F$, $x_2 \in F$, $x_2 = hx_1$, $h \in s(E)$, *then* r_{x_2} *and* $_h r_{x_1}$ *have the same germ at* x_2.

(c) *Let* $x_1 \in F$, $x \in U_{x_1}$. *There is an* $h \in H$ *such that* $hx = x_2 \in F$ *and* r_{x_2} *and* $_h r_{x_1}$ *have the same germ at* x_2.

Then Y *has a unique topology satisfying* 3.14 *and such that the sections* r_x *are continuous. In particular* $t : F \to Y$ *is a continuous section.*

PROOF. The proof is given in the language of presheaves. One could give it also by defining neighbourhood bases of the points of Y. Let U be an open subset of X. Define $R(U)$ to be the set of sections $r : U \to Y$ with the following property: For any $x \in U$ there is an $h \in H$ and an $x_1 \in F$ such that $hx_1 = x$ and r and $_h r_{x_1}$ have the same germ at x. The $R(U)$ obviously form a presheaf, satisfying the two Serre conditions.

Furthermore,

(i) If $r \in R(U)$ then $_h r \in R(hU)$

(ii) $r_x \in R(U_x)$ for $x \in F$