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# Lie Groups and Compact Groups

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# Preface

The purpose of these notes is twofold: to provide a quick self-contained introduction to the general theory of Lie groups and to give the structure of compact connected groups and Lie groups in terms of certain distinguished 'simple' Lie groups. With regards to the first aim, the notes can be used to provide a general introduction to the fundamentals of Lie groups or as a bridge to more advanced texts. In either case, experience has shown that they are suitable for postgraduate students and, at least the earlier chapters, for senior undergraduates. Concerning the second aim, the existing treatments of the structure results referred to above seem to be all from a fairly advanced point of view (cf. Pontrjagin [1] and Weil [1]). It is hoped that the present, more modern treatment makes these powerful results more generally accessible, in particular to those only wishing to use them as a tool.

The theory of Lie groups lies at the junction of the theories of differentiable manifolds, topological groups and Lie algebras. In keeping with current trends, when dealing with manifolds (and hence with Lie groups) a coordinate-free notation is used, thus removing the necessity for tedious juggling of indices and, hopefully, adding to the clarity and intuitiveness of the theory. In the case of Lie groups, particular emphasis is placed upon results and techniques which educe the interplay between a Lie group and its Lie algebra.

During the past few years a number of important results have been obtained in harmonic analysis on compact groups and compact Lie groups by using the structure of these groups ... the overall orientation of the following notes is to give full details of several of these structure results. The main theorem for Lie groups is that if G is a compact connected Lie group, then G is topologically isomorphic to

$$(G_0 \times G_1 \times \ldots \times G_m)/K,$$

where $G_0$ is the identity component of the centre of $G$, the $G_j$ $(j = 1, \ldots, m)$ are all the simple, connected, normal Lie subgroups of $G$, and $K$ is a finite subgroup of the centre of the product. As a corollary, a similar structure theorem is given in which the $G_j$ are also simply connected. This latter result is then generalised to arbitrary compact connected groups.

The decision on whether to include a particular result was based almost entirely on whether or not it was required for the proofs of the above structure theorems. This procedure accounted for the inclusion of most of the fundamental results and concepts in the theory of Lie groups; to round off the notes it only remained to add a few divertimenti such as the contents of Chapter 4 on the geometry of Lie groups or the list in Chapter 6 of necessary and sufficient conditions for a compact group to be Lie.

Chapter 1 contains results in the theory of analytic manifolds which are basic to the study of Lie groups. Chapter 2 begins the study of Lie groups and it is here that most of the fundamental concepts such as Lie algebras, left invariant vector fields, 1-parameter subgroups and the exponential map are introduced. In Chapter 3 the first deep result is presented; this is the Campbell-Baker-Hausdorff formula and it describes a relationship between the group structure of a Lie group and the algebraic structure of its Lie algebra. Chapter 4 introduces the notion of a geodesic on a Lie group and uses the resulting ideas to show that the exponential map is surjective whenever the Lie group is compact and connected. The correspondence between Lie subgroups of a Lie group and subalgebras of its Lie algebra is treated in Chapter 5. Chapter 6 begins with a list of conditions which are necessary and sufficient for a compact group to be Lie and ends with the structure results mentioned above. An appendix contains all the results on locally compact topological groups and their representations used in the body of the notes.

Further remarks, historical and motivational, on the contents of a chapter are given at the end of that chapter, along with related exercises. That a piece of theory is essential to a particular proof is no bar to it being included as an exercise if it is fairly straightforward or if it is fully treated in the literature.

I gave a course on some of the topics treated in these notes during 1973 at the Australian National University to an audience consisting mainly of postgraduate students, and then in 1974 at the University of New South Wales. These notes derive from these courses and in particular from duplicated notes of the earlier chapters. I am grateful to the people attending these courses for improvements of a number of arguments and in particular to Dr. Graham Wood for his reading of Chapter 1 and subsequent discussions. It was he who developed the local coordinate-free formula given in 1.3.2 and 1.3.3 for the Lie product of two analytic vector fields.

Finally, I feel that this preface would not be complete without some mention of the role of diagrams. Even though a large number of the concepts and results of manifolds and Lie groups have a strong pictorial or diagrammatic aspect, my experience is that diagrams in mathematics books are often of little value without a personal explanation. For this reason and because of widely varying preferences as to style, apart from several 'commutative arrow diagrams', none have been included here. However, without doubt they are valuable in developing an intuition in this area and the reader is strongly encouraged to experiment with them. Also some have found benefit in reformulating key results in terms of coordinates.

In the later chapters a number of substantial results are stated without proof since it is felt that to include them would take us too far afield in a set of lecture notes. However, the omitted proofs are all clearly presented in numerous standard texts to which detailed references are given. This also allows a clearer path to the structure theorems.

Kensington, 1976 J. F. P.