1·Analytic manifolds

This chapter contains the basic theory of analytic manifolds modelled on finite-dimensional real vector spaces. As promised, a coordinate-free approach will be used with emphasis on global definitions and properties. One of the reasons for including this chapter, instead of referring the reader to one or other of the numerous texts on manifolds, is to allow the reader to gain familiarity with this approach since it will permeate our whole treatment of Lie groups. Once one does away with coordinates it becomes obvious that large chunks of the theory of manifolds can be effortlessly generalised to manifolds modelled on infinite-dimensional spaces. We will have no need here of this degree of generality for reasons explained in the Notes at the end of the chapter, but the interested reader should consult the works of Lang, [1] and [2]. Since the theory of manifolds is one of the three legs on which the study of Lie groups stands, the other two being the theory of locally compact groups and the theory of Lie algebras, it is important that the ideas in this chapter, few though they may be, are well understood.

1.1 Manifolds and differentiability

1.1.1 **Manifolds.** Let M be a nonvoid Hausdorff topological space and E a real finite-dimensional vector space. If $\phi : U \to V$ is a homeomorphism between open subsets U and V of E and M respectively, then we say that ϕ is a chart on M. Also, if $p \in V$, then we say that ϕ is a chart about p. (Thanks to the infiltration of notions from category theory, it is now respectable to suppose that whenever a function is specified, its domain and codomain are automatically specified along with it. Hence there is no need to always explicitly write each function as a triple. We will adopt this convention here and immediately make use of it by supposing that whenever ϕ, ϕ_α and ϕ_β are charts, then their domains are U, U_α and U_β respectively, and their codomains (which in this case are also their ranges) are V, V_α and V_β respectively, unless otherwise specified.)

Suppose that ϕ_α is a chart on M for each α in some index set A. Then this collection, denoted $(\phi_\alpha : \alpha \in A)$, is called an atlas on M provided:

(i) each U_α is contained in the same finite-dimensional space, E say; and

(ii) the union of the V_αs is equal to M.

In this case we say that M is a manifold or M is a manifold modelled on E. (When we wish to be completely explicit we will say that the pair $(M, (\phi_\alpha : \alpha \in A))$ is a manifold. However, when no confusion seems possible, we will write only 'M is a manifold'.) The dimension of M as a manifold is defined as the dimension of E. Regarding the invariance of dimension, see Exercise 1.C(i).

It is obvious that every open subset U of a real finite-dimensional space E is a manifold when equipped with its identity map i. Henceforth, whenever we refer to such a set U as a manifold, its atlas will always be assumed to be $i : U \to U$. Less trivial examples of a manifold will be given in Chapter 2 after the definition of a Lie group.

Convention. It is easily seen that a Hausdorff topological space M can be equipped with a 0-dimensional atlas if and only if the topology of M is discrete. Thus, even though all the ensuing results on manifolds and Lie groups are valid for the 0-dimensional case, they are banal. Hence we will make the convention that the dimensions of all linear spaces, manifolds, and Lie groups are at least 1. In those cases when we want to emphasise that the dimension of a linear real space is n, we will often write it as R^n, where R denotes the real line. Generally, however, such finite-dimensional real spaces will be denoted by E or F.

1.1.2 Differentiable maps. The abstract definition of the derivative of a map between finite-dimensional vector spaces is the main point of departure from the classical approach to differentiable manifolds to one involving no explicit mention of coordinates. Given a function f from an open subset U of a finite-dimensional real space E into another such space F, then we say that f is differentiable at x in U if there exists a linear map $f'(x) : E \to F$ such that

$$(1.1.1) \qquad \lim_{\varepsilon \to 0} \varepsilon^{-1}(f(x + \varepsilon h) - f(x)) = f'(x)h$$

uniformly for h in any bounded subset of U (provided, of course, that $x + \varepsilon h \in U$). This is readily seen to be equivalent to the existence of a linear map $f'(x) : E \to F$ such that

$$(1.1.2) \qquad \lim_{h \to 0, h \neq 0} \frac{\|f(x+h) - f(x) - f'(x)h\|}{\|h\|} = 0 .$$

Here the norm is taken as any one of the equivalent norms which make the finite-dimensional space E into a Banach space. (See Edwards [1, Proposition 1.9.6].) Throughout the sequel, whenever the need for a topology on a finite-dimensional vector space arises, then it will always be taken to be the topology induced from such a norm.

Exercise 1.A collects together some of the elementary properties of this derivative, for example, the uniqueness of the linear map $f'(x)$.

If f is differentiable at each point of U, we say that f is <u>differentiable on</u> U. In this case we have the function

$$f' : U \to \hom(E, F),$$

where $f' : x \mapsto f'(x)$ and $\hom(E, F)$ is the linear space of linear maps from E into F. Continuing in this way it is clear that we may have higher order derivatives $f'' = (f')'$, $f''' = (f'')'$, and so on. In this way we arrive at the notion of a <u>smooth</u> function (at a point or on an open set) being a function which possesses derivatives of all orders (in a neighbourhood of the point or in the open set). Suppose that f is as above and that f'' exists on U, then f'' is a function from U into $\hom(E, \hom(E, F))$. As is customary, we identify this latter space in the canonical manner with $\hom^2(E \times E, F)$, the bilinear maps from $E \times E$ into F. In fact, throughout we adopt the convention that if $f^{(p)}$ exists on U, then its image space is $\hom^p(E \times \ldots \times E$ (p times), $F)$. This simplifies a number of expressions, including Taylor's expansion in 1.1.5 below.

If E, F and G are real finite-dimensional spaces and $f : E \to F$ and $g : F \to G$ are differentiable at x and $f(x)$ respectively, a classical result (included below in Exercise 1.A) states that $g \circ f$ is also differentiable at x and moreover:

$$(1.1.3) \qquad (g \circ f)'(x) = g'(f(x)) \circ f'(x).$$

1.1.3 **Remarks.** The notion of the derivative given above is often called the Fréchet derivative. For Banach spaces the study of this derivative forms Chapter VIII of Dieudonné [1], while Averbukh and Smolyanov [1] study this and related derivatives on topological vector spaces in general. For example, these latter authors show that in a certain sense the Fréchet derivative is the weakest type of differentiation for which the first order chain rule, formula (1.1.3) above, is valid for finite-dimensional spaces [1, p. 74].

1.1.4 **Maps from R.** When considering a differentiable map $f : \mathbf{R} \to E$, then $f'(x)$ satisfies

$$f'(x)(t) = f'(x)(1).t \text{ for each } t \text{ in } \mathbf{R}.$$

Thus $f'(x)$ is completely described by its value at 1 and we often write $f'(x)$ in place of $f'(x)(1)$. (This is precisely what happens in the classical case of functions from $\mathbf{R}$ into $\mathbf{R}$ where the derivative $f'(x)$ is taken to be a number as opposed to an operator.)

1.1.5 **Analytic functions.** Suppose that f is a smooth function from an open subset U of a real finite-dimensional space E into another such space F. Let x in U and y in E be such that $x + ty \in U$ for all $t \in [0, 1]$. If $y^{(m)}$ denotes the m-tuple $(y, \ldots, y)$, then

$$(1.1.4) \qquad f(x+y) = f(x) + \frac{1}{1!} f'(x)y + \ldots + \frac{1}{m!} f^{(m)}(x)y^{(m)} + R_{m+1}(y)$$

for each $m \in Z^+ = \{0, 1, 2, \ldots\}$, where the error term R_{m+1} satisfies $\lim_{y \to 0} R_{m+1}(y). \|y\|^{-m} = 0$. (See Exercise 1.D, where one particular version of the error term is described.) The sum (1.1.4) is called Taylor's formula of degree m.

Just as in the 1-dimensional case, we say that a smooth function $f : U \to F$ is (real) analytic on U if for each x in U there exists an open ball $B \subseteq U$ with centre x such that for all $z = x + y$ in B, the series

$$(1.1.5) \qquad \sum_{m=0}^{\infty} \frac{1}{m!} f^{(m)}(x)y^{(m)}$$

is absolutely convergent (that is, $\sum_m \frac{1}{m!} \|f^{(m)}(x)y^{(m)}\|$ is convergent, where the norm is that of F) and converges to $f(z)$. The function $f : U \to F$ is said to be analytic at x if it is analytic in some neighbourhood of x.

Examples. (i) If $f : E \to F$ and $g : F \to G$ are analytic at x and $f(x)$ respectively, then $g \circ f$ is analytic at x.

(ii) If $f : U \to F$ is analytic on U, an open subset of E, then $f^{(k)}$ is also analytic on U for each $k \in Z^+$ and its expansion at $x + y \in U$, where $x \in U$, is $\sum_m \frac{1}{m!} f^{(k+m)}(x)y^{(m)}$; in other words,

$$f^{(k)}(x+y)(u_1, \ldots, u_k) = \sum_{m=0}^{\infty} \frac{1}{m!} f^{(k+m)}(x)(\overbrace{y, \ldots, y}^{m \text{ terms}}, u_1, \ldots, u_k).$$

The validation of these two examples is left as an exercise for the interested reader.

(iii) Examples of smooth functions which are not analytic are well known. Even the absolute convergence of (1.1.5) in a ball is not sufficient to ensure the analyticity of the function at the centre of the ball concerned. For example, consider the smooth function $g : \mathbf{R} \to \mathbf{R}$ defined by

$$g(x) = \begin{cases} e^{-1/x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}.$$

It satisfies $g^{(m)}(0) = 0$ for all $m \in Z^+$ so that the series (1.1.5) is absolutely convergent for all $y \in \mathbf{R}$, but it only converges to $g(x)$ when $x = 0$.

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1.1.6 **Smooth atlases and manifolds.** An atlas $(\phi_\alpha : \alpha \in A)$ on the Hausdorff topological space M is said to be smooth if each of the functions $\phi_\beta^{-1} \circ \phi_\alpha$ is smooth on $\phi_\alpha^{-1}(V_\alpha \cap V_\beta)$. Such an atlas is said to be maximal if whenever U and V are open subsets of E and M respectively and $\phi : U \to V$ is a homeomorphism with the property that the functions

$$(1.1.6) \quad \phi_\alpha^{-1} \circ \phi : \phi^{-1}(V \cap V_\alpha) \to \phi_\alpha^{-1}(V \cap V_\alpha)$$

$$(1.1.7) \quad \phi^{-1} \circ \phi_\alpha : \phi_\alpha^{-1}(V \cap V_\alpha) \to \phi^{-1}(V \cap V_\alpha)$$

are smooth for each $\alpha \in A$, then $\phi \in (\phi_\alpha : \alpha \in A)$.

Lemma. Every smooth atlas on M is contained in a unique maximal smooth atlas.

Proof. If $(\phi_\alpha : \alpha \in A)$ is smooth on M, let $(\phi_{\alpha'} : \alpha' \in A')$ denote the collection of all maps ψ which are homeomorphisms between open subsets of E and M and which satisfy (1.1.6) and (1.1.7). This collection is an atlas with the desired properties. //

As a matter of terminology, a smooth atlas is said to generate the unique maximal smooth atlas which contains it.

Definition. A manifold $(M, (\phi_\alpha : \alpha \in A))$ is said to be smooth if the atlas $(\phi_\alpha : \alpha \in A)$ is both smooth and maximal.

In practice it is more usual to work with generating atlases rather than the corresponding maximal atlases since, as for the case of sub-bases in topology, most of the properties with which we are concerned are valid on a maximal atlas if valid on any of its generating atlases. Thus if we specify a smooth atlas $(\phi_\alpha : \alpha \in A)$ on M and then refer to $(M, (\phi_\alpha : \alpha \in A))$ as a smooth manifold, the precise meaning is that we are to take M equipped with the maximal smooth atlas generated by $(\phi_\alpha : \alpha \in A)$.

For example, if M is a real finite dimensional space equipped with its usual topology and $i : M \to M$ is the identity map, then $(M, \{i\})$ is a smooth manifold. This is simple enough but even here the maximal smooth atlas generated by i contains a superabundance of members. As an exercise describe them.

1.1.7 **Smooth maps.** If M and N are smooth manifolds with smooth atlases $(\phi_\alpha : \alpha \in A)$ and $(\psi_\beta : \beta \in B)$ respectively, we say that a map f from U, an open subset of M, into N is smooth if each of the maps $\psi_\beta^{-1} f \phi_\alpha$, defined on $\phi_\alpha^{-1}(V_\alpha \cap f^{-1}(V_\beta))$, is smooth.

In particular, f is smooth at the point x if and only if

(i) $\psi_\beta^{-1} f \phi_\alpha$ is smooth at $\phi_\alpha^{-1}(x)$ for every pair $(\phi_\alpha, \psi_\beta)$ satisfying

(ii) $x \in \operatorname{codom} \phi_\alpha$, $f(x) \in \operatorname{codom} \psi_\beta$.

However, because of the smoothness of the atlases involved, we need only consider the smoothness of (i) for any particular pair satisfying (ii). For example, suppose that (i) is satisfied by the pair $(\phi_\alpha, \psi_\beta)$ in (ii), and further suppose that $(\phi_{\alpha'}, \psi_{\beta'})$ is another pair satisfying (ii). Then certainly $\phi_\alpha^{-1}\phi_{\alpha'}$ and $\psi_{\beta'}^{-1}\psi_\beta$ are smooth at $\phi_{\alpha'}^{-1}(x)$ and $\psi_\beta^{-1}(f(x))$ respectively. Thus

$$\psi_{\beta'}^{-1} f \phi_{\alpha'} = (\psi_{\beta'}^{-1}\psi_\beta)(\psi_\beta^{-1} f \phi_\alpha)(\phi_\alpha^{-1}\phi_{\alpha'})$$

is smooth at $\phi_{\alpha'}^{-1}(x)$ showing that (i) is also satisfied by the pair $(\phi_{\alpha'}, \psi_{\beta'})$.

If M and N are smooth manifolds and if $f : M \to N$ is a homeomorphism such that f and f^{-1} are smooth, then f is said to be a diffeomorphism.

1.1.8 **Analytic manifolds.** If in the above definitions of smooth atlases, manifolds and maps between manifolds, we replace 'smooth' by 'analytic', then we arrive at the definitions of analytic atlases, manifolds and maps between manifolds. If a homeomorphism and its inverse are analytic, then the homeomorphism is said to be an analytic homeomorphism or even an analytic diffeomorphism.

1.1.9 **A condition for analyticity.** Let M and N be analytic manifolds with analytic atlases $(\phi_\alpha : \alpha \in A)$ and $(\psi_\beta : \beta \in B)$ respectively. Analogously to 1.1.7, a map $f : M \to N$ is analytic at x in M if and only if there exist charts ϕ_α about x and ψ_β about f(x) such that $\psi_\beta^{-1} f \phi_\alpha$ is analytic at x.

1.2 The tangent bundle

1.2.1 **The basic idea.** Let U be an open subset of a finite-dimensional real space E. If $\xi : (-\varepsilon, \varepsilon) \to U$, $\varepsilon > 0$, is an analytic curve satisfying $\xi(0) = p$, then, either by calculus or imagination, ξ has a tangent at p. Moreover, two analytic curves passing through p have the same tangent provided they have the same 'direction' and the same 'speed' at p. In mathematical terms, the tangent to ξ at p is defined as $\xi'(0)$ (or, $\xi'(0)(1)$), a vector in E. Thus the tangent space of U at p may be thought of as E and can be given a concrete realisation as the family of analytic curves, equivalent modulo their derivatives at p, passing through p.

Now suppose that M is an analytic manifold modelled on E. Let $\mathcal{C}_p(M)$ denote the set of analytic maps ξ from open neighbourhoods of 0 in **R** into M which satisfy $\xi(0) = p$. The derivative of ξ is not defined, but we circumvent this difficulty by considering the derivative of $\phi_\alpha^{-1}\xi$ at 0, where ϕ_α is a chart about p. Define an equivalence relation on $\mathcal{C}_p(M)$ by

$$(1.2.1) \quad \xi \sim \eta \text{ if } (\phi_\alpha^{-1}\xi)'(0) = (\phi_\alpha^{-1}\eta)'(0).$$

Let $[[\xi]]_p$ denote the class of curves equivalent to ξ (and note that if two curves are equivalent as in (1.2.1), then they are equivalent for each chart about p); the tangent space to M at p is defined as the set of all such equivalence classes.

Proceeding in the opposite direction, given any v in E and any analytic chart ϕ_α about p, can we always find a curve ξ in $\mathcal{C}_p(M)$ such that $(\phi_\alpha^{-1}\xi)'(0) = v$? The answer is 'yes' and forms part of Exercise 1.B.

Thus this fairly intuitive approach to the idea of a tangent space at a point of a manifold modelled on E shows that it is always isomorphic to E. Once this point is seen, it becomes notationally easier to proceed straight to E without any consideration of analytic curves. This will be done in the next subsection. A further approach to tangent spaces via 'derivation mappings' is contained in Exercise 1. F.

1. 2. 2 Tangent spaces. Suppose that $(M, (\phi_\alpha : \alpha \in A))$ is an analytic manifold and that p is a point on M. We define an equivalence relation on the pairs (ϕ_α, v), where ϕ_α is a chart about p and $v \in E$ by

$$(1.2.2) \quad (\phi_\alpha, v) \sim_p (\phi_\beta, w) \ \text{ if } \ (\phi_\beta^{-1}\phi_\alpha)'(\phi_\alpha^{-1}p)(v) = w.$$

This process may be thought of in the following way: corresponding to each chart about p we attach the space E to the point p, each pair of attachments then being identified under the equivalence relation (1. 2. 2). This, in effect, leaves us an object at p independent of all charts about p. We will see shortly that this object has a natural linear structure which makes it isomorphic with E. With this structure it is called the tangent space of M at p. Motivation for (1. 2. 2) comes by pursuing further the preceding discussion in 1. 2. 1 on analytic curves ξ on M with $\xi(0) = p$. It is reasonable to ask that the tangent at p corresponding to ξ depends on ξ only and not on different charts ϕ_α and ϕ_β, say, about p, so in some sense we want $(\phi_\alpha^{-1}\xi)'(0)$ and $(\phi_\beta^{-1}\xi)'(0)$ to be equivalent. They are, under (1. 2. 2)!

Before going further, we should verify that (1. 2. 2) is indeed an equivalence relation. To do this we must ascertain that it is (i) reflective, (ii) symmetric and (iii) transitive.

(i) It is immediate that $(\phi_\alpha, v) \sim_p (\phi_\alpha, v)$ and hence that the relation is reflexive.

(ii) Suppose that $(\phi_\alpha, v) \sim_p (\phi_\beta, w)$ and hence $(\phi_\beta^{-1}\phi_\alpha)'(\phi_\alpha^{-1}p)v = w$; the relation is symmetric provided $(\phi_\alpha^{-1}\phi_\beta)'(\phi_\beta^{-1}p)w = v$. Now

$$\begin{aligned}(\phi_\alpha^{-1}\phi_\beta)'(\phi_\beta^{-1}p)w &= (\phi_\alpha^{-1}\phi_\beta)'(\phi_\beta^{-1}p)(\phi_\beta^{-1}\phi_\alpha)'(\phi_\alpha^{-1}p)v \\ &= (\phi_\alpha^{-1}\phi_\beta\phi_\beta^{-1}\phi_\alpha)'(\phi_\alpha^{-1}p)v\end{aligned}$$

(using the chain rule (1. 1. 3))

$$= (\mathrm{Id})'(\phi_\alpha^{-1}p)v$$

$$= v,$$

as required.

(iii) If $(\phi_\alpha, v) \sim_p (\phi_\beta, w)$ and $(\phi_\beta, w) \sim_p (\phi_\gamma, u)$, then the transitivity of the relation is established provided $(\phi_\alpha, v) \sim_p (\phi_\gamma, u)$, that is, provided $(\phi_\gamma^{-1}\phi_\alpha)'(\phi_\alpha^{-1}p)v = u$. Making use of the chain rule yields

$$(\phi_\gamma^{-1}\phi_\alpha)'(\phi_\alpha^{-1}p)v = (\phi_\gamma^{-1}\phi_\beta\phi_\beta^{-1}\phi_\alpha)'(\phi_\alpha^{-1}p)v$$

$$= (\phi_\gamma^{-1}\phi_\beta)'(\phi_\beta^{-1}p)(\phi_\beta^{-1}\phi_\alpha)'(\phi_\alpha^{-1}p)v$$

$$= (\phi_\gamma^{-1}\phi_\beta)'(\phi_\beta^{-1}p)w$$

$$= u,$$

the sought-after relation.

Denote the class equivalent to (ϕ, v) at p by $[\phi, v]_p$. The set of these equivalence classes is given a linear structure by defining

$$\lambda[\phi_\alpha, v]_p = [\phi_\alpha, \lambda v]_p \quad \text{for } \lambda \text{ in } \mathbb{R}, \text{ and}$$

$$[\phi_\alpha, v]_p + [\phi_\beta, w]_p = [\phi_\beta, (\phi_\beta^{-1}\phi_\alpha)'(\phi_\alpha^{-1}p)v + w]_p.$$

(For complete rigour, it should be checked that these linear operations are well-defined but we will take the easy way and refer the reader to Exercise 1. B.)

The set of all equivalence classes under (1. 2. 2) with the linear structure defined above will be denoted by $T_p(M)$ and is called the <u>tangent space of</u> M <u>at</u> p. Clearly it is linearly isomorphic with E. Since, in effect, we are adjoining a copy of E to each point of M, if p and q are distinct points in M, the question of whether $T_p(M)$ and $T_q(M)$ overlap does not arise.

1. 2. 3 **The tangent bundle.** Suppose that $(M, (\phi_\alpha : \alpha \in A))$ is an analytic manifold modelled on E; the <u>tangent bundle</u> of M is defined as the (disjoint) union of the tangent spaces $T_p(M)$, p running over M, and will be denoted by $T(M)$.