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I. M. James
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1·Introduction: algebra versus topology

There are three families of Stiefel manifolds, the real, the complex and the quaternionic. Readers of these notes may already be familiar with the account of their basic properties to be found in standard texts such as Steenrod [133] and Steenrod-Epstein [134]; a summary is given in §2 below. In this introduction we shall only be dealing with the real family, which is undoubtedly the most interesting. Some of the real Stiefel manifolds have particular topological properties, due to the existence of certain constructions which are algebraic in origin. Our aim is to try and understand, from the topological point of view, why some of them have these properties while others do not.

The notation we use is fairly standard. Thus R^m denotes euclidean m-space $(m = 0, 1, \ldots)$ with the usual embedding of R^m in R^{m+1}. The vectors $v \in R^m$ such that $|v| \leq 1$ form the unit ball B^m and those such that $|v| = 1$ form the unit sphere S^{m-1}. The projective space P^{m-1} is obtained from S^{m-1} by identifying v with $-v$ for all $v \in S^{m-1}$. The group of orthogonal transformations of R^m is denoted by O_m. Thus $P^{m-1} \subset P^m$ and $O_m \subset O_{m+1}$, in the usual way. Unless it is necessary to be more specific the basepoint in any space is denoted by e; orientation conventions are as in [64], and $\iota_m \in \pi_m(S^m)$ denotes the class of the identity self-map.

Following Stiefel [136] and many others let $V_{n,k}$, where $1 \leq k \leq n$, denote the manifold of orthonormal k-frames in R^n. Elements of $V_{n,k}$ correspond, in an obvious way, to norm-preserving linear transformations of R^k into R^n. The orthogonal group O_k acts on $V_{n,k}$ by pre-composition, while the orthogonal group O_n acts on $V_{n,k}$ by post-composition. The latter action is transitive and enables $V_{n,k}$ to be identified with the factor space of O_n by O_{n-k}. For $k < n$ the rotation group can be used instead of the full orthogonal group.

If we pre- or post-compose with a rotation we obtain a self-map

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of $V_{n,k}$ in the homotopy class of the identity. If we pre-compose by a non-rotation we obtain a self-map of homotopy class λ, say; if we post-compose by a non-rotation we obtain a self-map of homotopy class μ, say. In the semigroup of homotopy classes of self-maps of $V_{n,k}$ these canonical classes satisfy the relations

$$(1.1) \quad \lambda^2 = 1 = \mu^2, \quad \lambda\mu = \mu\lambda, \quad \lambda^k = \mu^n.$$

To prove the last of these, represent elements of $V_{n,k}$ by matrices with k columns - the vectors of the k-frame - and n rows. The class λ includes the self-maps which change the sign of any column. The class μ includes the self-maps which change the sign of any row. Since changing the sign of all the columns has the same effect as changing the sign of all the rows we obtain $\lambda^k = \mu^n$, as asserted. Note that $\lambda = 1$ if n is even and k odd, while $\mu = 1$ if n is odd and k even. In some applications it is the class $\xi = \lambda\mu$ which is important; note that $\xi = 1$ if n and k are both odd.

We can fibre $V_{n,k}$ over $V_{n,1} = S^{n-1}$ by taking one vector (say the last) from each k-frame. A cross-section $f : S^{n-1} \to V_{n,k}$ associates with each point $v \in S^{n-1}$ an orthonormal k-frame $(v_1, \ldots, v_{k-1}, v)$. Thus $(v_1, \ldots, v_{k-1}) = g(v)$, say, is an orthonormal (k - 1)-frame; we refer to $g : S^{n-1} \to V_{n,k-1}$ as the projection of f. We can always regard $(v_1, \ldots, v_{k-1})$ as a (k - 1)-frame of tangents to S^{n-1} at the point v. Hence a cross-section of $V_{n,k}$ over S^{n-1} is equivalent to an (orthonormal) (k - 1)-field on S^{n-1}, i.e. a field of orthonormal tangent (k - 1)-frames. Any such (k - 1)-field spans a field of tangent (k - 1)-planes. Conversely Steenrod has shown, in §27 of [133], that if S^{n-1} admits a field of tangent (k - 1)-planes and $2k \leq n + 1$ then S^{n-1} admits a (k - 1)-field. This does not mean, however, that every field of tangent (k - 1)-planes can be spanned by a (k - 1)-field (see [62]).

For what values of n and k does $V_{n,k}$ admit a cross-section, over S^{n-1}? Take $k = 2$, for example. We need to find a self-map g of S^{n-1} such that g(v) is orthogonal to v, for all $v \in S^{n-1}$. When n is even, say $n = 2m$, we can regard v as a complex m-vector, rather than a real 2m-vector, and define g through multiplication by i. In terms of coordinates, if $v = (x_0, x_1, \ldots, x_{2m-2}, x_{2m-1})$ then

$g(v) = (-x_1, x_0, \ldots, -x_{2m-1}, x_{2m-2})$. Conversely, suppose that g exists, with $g(v)$ orthogonal to v. Then

$$h_t(v) = v \cos \pi t + g(v) \sin \pi t \qquad (0 \leq t \leq 1)$$

defines a homotopy between the identity on S^{n-1} and the antipodal map. Since the degree of the latter is $(-1)^n$ it follows at once that n is even. Thus $V_{n,2}$ admits a cross-section if and only if n is even.

Given k, we can construct cross-sections of $V_{n,k}$ for suitable values of n as follows. Consider the Clifford algebra $C_m (m = 0, 1, \ldots)$ generated by an anticommuting set of elements $(e_1, \ldots, e_m)$ such that

$$e_1^2 = \ldots = e_m^2 = -1.$$

Thus $C_0 = R$, the real numbers; $C_1 = C$, the complex numbers; and $C_2 = H$, the quaternions. The next five Clifford algebras are easily shown to be

$$H \oplus H, \ H(2), \ C(4), \ R(8), \ R(8) \oplus R(8),$$

where $A(q)$, for any algebra A and positive integer q, denotes the q^{th} order matrix algebra over A. Moreover (see [9], for example) the matrix algebra $C_m(16)$ of order 16 over C_m is isomorphic to C_{m+8}. Thus all the Clifford algebras can be expressed in terms of matrix algebras over R, C or H.

Let $\sigma(k)$ denote the number of integers s in the range $0 < s < k$ such that $s \equiv 0, 1, 2$ or $4 \bmod 8$. Clearly R^n can be represented as a C_{k-1}-module whenever $n \equiv 0 \bmod a_k$, where $a_k = 2^{\sigma(k)}$. Any such representation can be orthogonalized, in the usual way, so that the generators $e_1, \ldots, e_{k-1}$ correspond to orthogonal transformations, and then a cross-section $f : S^{n-1} \to V_{n,k}$ is given by

$$f(v) = (e_1 . v, \ldots, e_{k-1} . v, v) \qquad (v \in S^{n-1}).$$

The existence of these Clifford cross-sections was noted by Eckmann [38], with reference to the algebraic results of Hurwitz [60] and Radon [118]. We give an example, due to Zvengrowski, of a Clifford cross-section of

$V_{16,9}$ (the first eight column vectors are tangent to S^{15} at the points given by the last).

$$\begin{array}{ccccccccc}
x_8 & -x_7 & -x_6 & -x_5 & -x_4 & -x_3 & -x_2 & -x_1 & x_0 \\
-x_9 & x_6 & -x_7 & -x_4 & x_5 & -x_2 & x_3 & x_0 & x_1 \\
-x_{10} & -x_5 & -x_4 & x_7 & x_6 & x_1 & x_0 & -x_3 & x_2 \\
-x_{11} & -x_4 & x_5 & -x_6 & x_7 & x_0 & -x_1 & x_2 & x_3 \\
-x_{12} & x_3 & x_2 & x_1 & x_0 & -x_7 & -x_6 & -x_5 & x_4 \\
-x_{13} & x_2 & -x_3 & x_0 & -x_1 & x_6 & -x_7 & x_4 & x_5 \\
-x_{14} & -x_1 & x_0 & x_3 & -x_2 & -x_5 & x_6 & x_7 & x_6 \\
-x_{15} & x_0 & x_1 & -x_2 & -x_3 & x_4 & x_5 & -x_6 & x_7 \\
-x_0 & -x_{15} & -x_{14} & -x_{13} & -x_{12} & -x_{11} & -x_{10} & -x_9 & x_8 \\
x_1 & -x_{14} & x_{15} & x_{12} & -x_{13} & x_{10} & -x_{11} & x_8 & x_9 \\
x_2 & x_{13} & x_{12} & -x_{15} & -x_{14} & -x_9 & x_8 & x_{11} & x_{10} \\
x_3 & x_{12} & -x_{13} & x_{14} & -x_{15} & x_8 & x_9 & -x_{10} & x_{11} \\
x_4 & -x_{11} & -x_{10} & -x_9 & x_8 & x_{15} & x_{14} & x_{13} & x_{12} \\
x_5 & -x_{10} & x_{11} & x_8 & x_9 & -x_{14} & x_{15} & -x_{12} & x_{13} \\
x_6 & x_9 & x_8 & -x_{11} & x_{10} & x_{13} & -x_{12} & -x_{15} & x_{14} \\
x_7 & x_8 & -x_9 & x_{10} & x_{11} & -x_{12} & -x_{13} & x_{14} & x_{15}
\end{array}$$

It was Adams [3] who finally proved the long-conjectured

Theorem (1.2). The Stiefel manifold $V_{n,k}$ admits a cross-section, over S^{n-1}, if and only if $n \equiv 0 \bmod a_k$.

Sufficiency we have already established. Necessity is trivial for $k = 1$ and true for $k = 2$, as we have seen. For higher values of k various results were obtained by G. W. Whitehead [153], N. E. Steenrod and J. H. C. Whitehead [135], amongst others. To indicate the kind of methods used in this subject we shall now give the proof of (1.2) in case (i) $k - 1$ is a power two or (ii) $k \not\equiv 3 \bmod 8$. In particular we prove (1.2) for all $k \leq 10$. The remaining cases are more difficult and will be dealt with later.

The Stiefel manifold $V_{n,k}$ contains a subspace $P_{n,k}$ which plays a major role in what follows. To define $P_{n,k}$, first consider the real projective $(n-1)$-space $P^{n-1} = S^{n-1}/Z_2$. Any point $\pm x \in P^{n-1}$,

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where $x = (x_1, \ldots, x_n)$, determines a matrix

$$\| \delta_{ij} - 2x_i x_j \| \quad (i = n - k + 1, \ldots, n;\ j = 1, \ldots, n).$$

The k column vectors of this matrix constitute an orthonormal k-frame in R^n, i.e. an element of $V_{n,k}$. All points of the subspace $P^{n-k-1} \subset P^{n-1}$ spanned by the first $n - k$ coordinates determine the same element of $V_{n,k}$. We define $P_{n,k}$ to be the space P^{n-1}/P^{n-k-1} obtained from P^{n-1} by collapsing P^{n-k-1} to a point and regard $P_{n,k}$ as a subspace of $V_{n,k}$ under the embedding just described. When $k = n$ we interpret $P_{n,n}$ as the space obtained from P^{n-1} by adjoining a point corresponding to the identity matrix. Notice that $P_{n,1} = V_{n,1}$. In §3 below we shall prove

Proposition (1.3). The pair $(V_{n,k}, P_{n,k})$ is $(2n - 2k)$-connected.

In fact the pair can be given CW-structure so that $V_{n,k}$ is obtained from $P_{n,k}$ by attaching cells of dimension $2n - 2k + 1$ and higher. Now let S denote the suspension functor. A simple geometric construction, as follows, enables us to prove

Proposition (1.4). If $V_{n,k}$ has a cross-section then $S^n P_{m,k}$ has the same homotopy type as $P_{m+n,k}$ for all $m \geq k$.

Let $f : S^{n-1} \to V_{n,k}$ be a cross-section and let f_v, for $v \in S^{n-1}$, denote the norm-preserving transformation $R^k \to R^n$ corresponding to $f(v)$. Consider the map

$$\theta : B^n \times R^{m-k} \times R^k \to R^{m+n-k} \times R^k$$

which is given by

$$\theta(tv, y, z) = (y, tf_v(z), (1 - t^2)^{\frac{1}{2}} z)$$

where $0 \leq t \leq 1$ and $y \in R^{m-k}$, $z \in R^k$. Since $\theta(tv, -y, -z) = -\theta(tv, y, z)$, $|\theta(tv, y, z)| = |(y, z)|$, it follows that θ induces a map

$$(B^n \times P^{m-1}, B^n \times P^{m-k-1} \cup S^{n-1} \times P^{m-1}) \to (P^{m+n-1}, P^{m+n-k-1}),$$

and hence a map

$$\phi : (B^n/S^{n-1}) \wedge (P^{m-1}/P^{m-k-1}) \to (P^{m+n-1}/P^{m+n-k-1})$$

where $\wedge$ denotes the smash product. If f is a Clifford cross-section then ϕ is a homeomorphism. In the general case it can easily be shown (see §6) that ϕ induces an isomorphism in homology and hence is a homotopy equivalence, by the theorem of J. H. C. Whitehead [159].

Let us now see what information can be extracted from (1.4) by using the Steenrod squares in mod 2 cohomology. Recall that

$$H^*(P^{n-1}) = Z_2[a] \bmod a^n,$$

where a generates $H^1(P^{n-1})$, and that

$$Sq^i a^j = \binom{j}{i} a^{i+j},$$

by the Cartan product formula. From the cohomology exact sequence of the cofibration

$$P^{n-k-1} \to P^{n-1} \to P_{n,k}$$

we see that $\tilde{H}^r(P_{n,k})$, for $n - k \leq r < n$, is generated by an element a_r, where

$$(1.5) \quad Sq^i a_j = \binom{j}{i} a_{i+j}$$

for $j \geq n - k$ and $i + j < n$. With (1.4) in mind we prove

Proposition (1.6). Given n and k suppose that $S^n P_{m,k}$ and $P_{m+n,k}$ have the same homotopy type for all $m > k$. If $k = 2^s + 1$, for some s, then $n \equiv 0 \bmod 2^{s+1}$.

Choose $m > k$ so that $m \equiv k \bmod 2^{s+1}$. Then $Sq^i H^{m-k}(P_{m,k}) = 0$, by (1.5), for all $i > 0$. If n is an odd multiple of 2^r, where $r \leq s$, then $Sq^i H^{m+n-k}(P_{m+n,k}) \neq 0$ for $i = 2^r$, hence $S^n P_{m,k}$ and $P_{m+n,k}$ are not of the same homotopy type, since Sq^i commutes with suspension. This contradiction establishes (1.6) and hence, using (1.4), proves (1.2) when k - 1 is a power of two. The original argument of Steenrod and Whitehead is similar, except that (1.3) is used instead of (1.4).

Let us now replace cohomology by the functor $\tilde{K}_R$ formed from real vector bundles over a given space. Recall (see [9]) that $\tilde{K}_R(P^{n-1})$ is cyclic of order a_n with generator $\alpha = [L] - 1$, where L denotes the Hopf line bundle over P^{n-1}, and that $L^2 = L \otimes L$ is trivial. For any integer t the Adams operation ψ^t is defined, as in [3], and has the property that $\psi^t[L] = [L^t]$. Hence $\psi^t\alpha = 0$ or α according as t is even or odd. Just as in cohomology the exact sequence of the cofibration enables $\tilde{K}_R(P_{n,k})$ to be calculated. Provided $n \not\equiv k \bmod 4$ we find that $\tilde{K}_R(P_{n,k})$ can be identified with the subgroup of $\tilde{K}_R(P^{n-1})$ generated by $a_{n-k}\alpha$; when $n \equiv k \bmod 4$ there is an extra summand which complicates matters. Moreover $\psi^t = 0$ or 1 according as t is even or odd.

Let $\tau(k)$ denote the number of integers s in the range $0 < s < k$ such that $s \equiv 0, 1, 3$ or $5 \bmod 8$. Thus $\tau(k) = \sigma(k) - 1$ for $k \equiv \pm 3 \bmod 8$, and $\tau(k) = \sigma(k)$ otherwise. We prove

Proposition (1. 7). Given $n \equiv 0 \bmod 8$ and k, suppose that $S^nP_{m,k}$ and $P_{m+n,k}$ have the same homotopy type for all $m > k$. Then n is divisible by $2^{\tau(k)}$.

Choose $m > k$ so that $m \not\equiv k \bmod 4$ and write $\sigma(m) - \sigma(m-k+1) = f$. Recall that $\psi^t(S^*)^n = t^{n/2}(S^*)^n\psi^t$, for all values of t, where

$$(S^*)^n : \tilde{K}_R(P_{m,k}) \approx \tilde{K}_R(S^nP_{m,k}).$$

Let t be odd. Then $\psi^t = 1$ in the domain, as we have seen, and so $\psi^t = t^{n/2}$ in the codomain. On the other hand $\psi^t = 1$ in $\tilde{K}_R(P_{m+n,k})$. Since all these groups are cyclic of order 2^f this implies that

$$(1.\ 8) \qquad t^{n/2} \equiv 1 \bmod 2^f.$$

However if n is an odd multiple of 2^{e-2}, for any $e \geq 2$, then

$$(1.\ 9) \qquad 3^{n/2} - 1 \equiv 2^{e-1} \bmod 2^e,$$

by an elementary calculation as in §8 of [3]. Putting $t = 3$ we obtain an immediate contradiction unless n is an even multiple of 2^{f-2}. However m can be chosen, with $m \not\equiv k \bmod 4$, so that $f - 1 = 2^{\tau(k)}$, and so (1. 7) is proved.

To obtain (1.2) for $k \not\equiv \pm 3 \bmod 8$ we use (1.6), with (1.4), to deal with the cases $k \leq 4$ and to show that $n \equiv 0 \bmod 8$ when $k \geq 5$; then we use (1.7), with (1.4), to complete the proof. The original proof of (1.2) by Adams is similar, except that (1.3) and other results are used instead of (1.4).

Not every cross-section is homotopic to a Clifford cross-section, as can easily be seen, but a recent result of Milgram and Zvengrowski [111] is of interest here. A cross-section $f : S^{n-1} \to V_{n,k}$ is said to be skew if $f(v) = (v_1, \ldots, v_k)$ implies that $f(-v) = (-v_1, \ldots, -v_k)$. For example, Clifford cross-sections have this property. Milgram and Zvengrowski show that every cross-section is homotopic to a skew cross-section.

Another kind of cross-section is as follows. Consider the self-map T of $V_{n,k}$ which changes the sign of the last vector in each k-frame. Thus T is the antipodal map in the case of $V_{n,1} = S^{n-1}$. Let us say that a cross-section $f : S^{n-1} \to V_{n,k}$ is homotopy-equivariant if $Tf \simeq fT$. The case $k = 1$ is trivial. When $k \geq 2$ the condition can be taken as $Tf \simeq f$, since no cross-section exists unless n is even. If k is odd and n is even then $T \simeq 1$ on $V_{n,k}$, by (1.1). Hence the interest resides in the case k even. Notice that a cross-section of $V_{n,k+1}$ determines a homotopy-equivariant cross-section of $V_{n,k}$. For let $f_1, \ldots, f_k : S^{n-1} \to S^{n-1}$ be the first k components of a cross-section of $V_{n,k+1}$, and write $h_t(v) = (f_1 v, \ldots, f_{k-1} v, v \cos \pi t + f_k v \sin \pi t)$. Then h_0 is a cross-section of $V_{n,k}$ such that $h_0 \simeq h_1 = Th_0$. In §8 and §9 below we shall prove

Theorem (1.10). There exists a homotopy-equivariant cross-section of $V_{n,k}$ if and only if $n \equiv 0 \bmod \hat{a}_k$, where $\hat{a}_k = a_{k+1} = 2a_k$ for $k = 2$ or $k \equiv 0 \bmod 4$, and $\hat{a}_k = a_k$ otherwise.

Let us now turn to some problems which have not yet been solved. By general theory (see [133]) $V_{n,k}$ is trivial as a fibre bundle over S^{n-1} if and only if the associated principal bundle $V_{n,n} = O_n$ admits a cross-section, i.e. if and only if $n = 2$, 4 or 8. Thus $V_{4,k}$ $(k \leq 4)$ and $V_{8,k}$ $(k \leq 8)$ are trivial as fibre bundles. For triviality in the sense of fibre homotopy type, however, nothing is known beyond

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Theorem (1.11). *If* $V_{n,k}$ *is trivial as a fibre space over* S^{n-1} *then* $n = 2^r$ *for some* $r \geq \sigma(k)$. *Furthermore if* k *is even then* $n = 2$, 4 *or* 8.

The proof will be given in §20 below. It is tempting to conjecture that $V_{n,k}$ is non-trivial as a fibre space if it is non-trivial as a fibre bundle: the first unsettled case is that of $V_{16,3}$. As Scheerer [124] has pointed out the solution to this problem is important for the homotopy classification of Hopf homogeneous spaces.

Another unsolved problem concerns the self-map T of $V_{n,k}$ which changes the sign of the last vector in each k-frame. Let us say that $V_{n,k}$ is *neutral* (elsewhere *row-simple*) if $\lambda = 1$, where λ denotes the homotopy class of T, as before. Thus $V_{n,k}$ is neutral, by (1.1), whenever n is even and k odd. Moreover $V_{n,k}$ is neutral when $n = 3$ or 7 and k is even, since then $V_{n,k}$ is a retract of $V_{n+1,k+1}$, as remarked above. In §21 below we shall prove

Theorem (1.12). *Let* n *be odd and* k *even. If* $V_{n,k}$ *is neutral then either* $n + 1$ *or* $k - n + 1$ *is divisible by* 2^t, *where* t *denotes the least integer such that* $2^t > k$.

This gives no information when $k = 2$. However, in §22 we shall prove

Theorem (1.13). *Let* n *be odd. Then* $V_{n,2}$ *is neutral if and only if the Whitehead square* $w_n \in \pi_{2n-1}(S^n)$ *can be halved.*

Here w_n denotes the Whitehead product of the generator $\iota_n \in \pi_n(S^n)$ with itself. This vanishes, as is well known, if and only if $n = 1$, 3 or 7. Toda [144] has shown that w_{15} can be halved and Mahowald, in unpublished work, that w_{31} can be halved. It is not difficult to show that w_n $(n > 2)$ cannot be halved unless $n + 1$ is a power of two: in §23 below we shall prove

Theorem (1.14). *Let* n *be odd and let* $n \geq 2k - 2$, *where* $k = 2$, 4 *or* 8. *If* $V_{n,k}$ *is neutral then* $n + 1$ *is a power of two.*

It seems reasonable to conjecture that (1.14) is true for all even values of k.

Finally let us take another look at the problem of the existence of cross-sections. Suppose that we fibre $V_{n,k}$ over $V_{n,k-1}$ by taking the last $k - 1$ vectors of each k-frame to form a $(k - 1)$-frame. Any map $f : V_{n,k-1} \to V_{n,k}$ determines a map $g : V_{n,k-1} \to S^{n-1}$, by taking the first vector of each k-frame, and f is a cross-section if and only if the vector $g(v_1, \ldots, v_{k-1})$ is orthogonal to $v_1, \ldots, v_{k-1}$ for every orthonormal $(k - 1)$-frame $(v_1, \ldots, v_{k-1})$ in n-space. When $n = 3$ or 7 and $k = 3$ such a map g can be defined as follows. Elements of R^{n+1} can be regarded as quaternions when $n = 3$, as Cayley numbers when $n = 7$. Moreover the pure elements of the algebra (i. e. those with real part zero) determine a subspace which we identify with R^n. If u and v are pure then uv is orthogonal to both u and v; moreover uv is pure when u and v are themselves orthogonal. Hence a map g with the desired properties is defined by $g(u, v) = uv$. Thus $V_{n,3}$ admits a cross-section over $V_{n,2}$ for $n = 3$ or 7.

Conversely, if $V_{n,3}$ admits a cross-section over $V_{n,2}$ then S^n is an H-space and so $n = 3$ or 7, by the main theorem of Adams [1]. To see this, consider the unit ball $B^n \subset R^n$, of which S^{n-1} is the boundary, and the sphere S^n, of which S^{n-1} is the equator and $\pm e$, say, the poles. Given a cross-section $f : V_{n,2} \to V_{n,3}$ with projection $g : V_{n,2} \to S^{n-1}$, let $g' : B^n \times B^n \to B^n$ denote the map defined by

$$g'(au, bv) = ab \sin\theta \, g(u, (v - u\cos\theta)/\sin\theta),$$

where $u, v \in S^{n-1}$ and $a, b \in I = [0, 1]$, also $\cos\theta = u.v$, the inner product, for $0 \leq \theta \leq \pi$. Now let $h : S^n \times S^n \to S^n$ be defined by $h(\alpha e + x, \beta e + y) = (\alpha\beta e - x.y + \alpha y + \beta x + g'(x, y)$, where $x, y \in B^n$ and $-1 \leq \alpha, \beta \leq 1$. Clearly $h(\alpha e + x, e) = \alpha e + x$, $h(e, \beta e + y) = \beta e + y$, and so h constitutes an H-structure on S^n. Of course this construction is modelled on the formulae for quaternionic and Cayley multiplication contained in the previous paragraph. Summing up, we have proved

Theorem (1.15). <u>There exists a cross-section of</u> $V_{n,3}$ <u>over</u> $V_{n,2}$ <u>if and only if</u> $n = 3$ <u>or</u> 7.

Cross-sections of $V_{8,4}$ over $V_{8,3}$ have been exhibited by