Elementary results

Definition. A group of automorphisms of a group P stabilizes a chain $P = P_0 \supseteqq P_1 \supseteqq \ldots \supseteqq P_n = 1$ if $[A, P_i] \subseteqq P_{i+1}$, $i = 0, \ldots, n-1$. Here $[a, x] = x^{-a}x$ for $x \in P$, $a \in A$.

Theorem 0.1. If a group of automorphisms A of a π-group P stabilizes a chain $P \supseteqq P_1 \supseteqq \ldots \supseteqq P_n = 1$, then A is a π-group.

Proof. Suppose $a \in A$ is a π'-automorphism of P. Clearly, by induction we may assume that $[a, P_1] = 1$. Then if $x \in P$, $x^a = xy$ where $y \in P_1$, since $[a, P] \subseteqq P_1$.

It follows that $x^{a^2} = xy^2, \ldots, x^{a^{|a|}} = xy^{|a|} = x$.

Since y is a π-element while a is a π'-element, we have that $y = 1$ and $[a, P] = 1$. Thus $a = 1$ and A is a π-group. //

Corollary 0.2. If A is a π'-group of automorphisms of a π-group P such that $[P, A, A] = 1$, then $[P, A] = 1$ and so $A = 1$.

Proof. A stabilizes the chain $P \supseteqq [P, A] \supseteqq [P, A, A] = 1$. //

Lemma 0.3. Let A be a π'-group of automorphisms of a π-group P. Let Q be an A-invariant normal subgroup of P. Then $C_{P/Q}(A) = C_P(A)Q/Q$.

Proof. Clearly $C_P(A)Q/Q \subseteqq C_{P/Q}(A)$.

Suppose now that xQ is a coset of Q in P which is fixed by A. Let QA act as a group of permutations on xQ where A acts in the obvious way and Q acts by multiplication on the right. Then QA acts transitively on xQ since Q does. Let A_1 be the stabilizer of a point. Then $|A_1| = |QA|/|xQ| = |A|$.

By the Schur Zassenhaus-Feit-Thompson Theorem A_1 is conjugate to A. Thus there exists $y \in xQ$ such that $y \in C_P(A)$. //

Remark. Note that in every application of 0.3, 0.4 in these notes, it will be known a priori that at least one of A or P is solvable. Hence the Feit-Thompson Theorem will not be required in the applications of 0.3, 0.4 here.

Corollary 0.4. Let P be a π-group, A a π'-group of automorphisms of P. Then $P = [P, A]C_P(A)$.

Proof. $[P, A] \subseteq P$ and is A-invariant. Also A centralizes $P/[P, A]$. //

Corollary 0.5. If A is a π'-group of automorphisms of an abelian π-group P, then $P = C_P(A) \oplus [P, A]$.

Proof. We show that $C_P(A) \cap [P, A] = 0$, writing P additively. Let θ be the endomorphism of P defined by $\theta = \frac{1}{|A|} \sum_{a \in A} a$. Clearly $b\theta = \theta b = \theta$ for all $b \in A$. Thus $\theta^2 = \theta$.

Since $P\theta \cap \ker \theta = 0$, $P = P\theta \oplus \ker \theta$.

Now if $x \in C_P(A)$, then $x\theta = \frac{1}{|A|} \sum_{a \in A} xa = x$ and so $C_P(A) \subseteq P\theta$. Finally if $[x, a] \in [P, A]$, then $(-x + xa)\theta = -x\theta + x\theta = 0$ and so $[x, a] \in \ker \theta$. Thus $C_P(A) \cap [P, A] = 0$. The result follows from 0.4. //

Lemma 0.6. (Thompson) Any p-group P contains a characteristic subgroup C such that

(a) $cl\ C \leq 2$ and $C/Z(C)$ is elementary;

(b) $[P, C] \subseteq Z(C)$;

(c) $C_P(C) = Z(C)$;

(d) Any automorphism $a \neq 1$ of order prime to p acts non-trivially on C.

Proof. We first show that (c) and C char P together ensure (d). For suppose that $[a, C] = 1$, where a is our given p'-automorphism. Then

$$[a, C, P] = 1.$$

Also

$$[C, P, a] \subseteqq [C, a] = 1.$$

Thus

$$[P, a, C] = 1$$
$$[P, a] \subseteqq C_P(C) = Z(C) \text{ by (c).}$$

Hence

$$[P, a, a] = 1. \text{ By 0.2, } [P, a] = 1.$$

To show the existence of C we proceed as follows. First if any subgroup $A \in \mathcal{SCN}(P)$ is characteristic in P, then take $A = C$. Clearly (a), (b), (c) hold. Hence we may suppose that no maximal abelian subgroup of P is characteristic. Let D be a maximal characteristic abelian subgroup of P. Clearly $C_P(D) \supset D$ and $C_P(D)$ char P. Let

$$C/D = \Omega_1(Z(P/D)) \cap C_P(D)/D.$$

Clearly $C \supset D$, and C is a characteristic subgroup of P. Since $D \subseteqq Z(C)$ and $Z(C)$ is an abelian characteristic subgroup of P, maximality of D ensures that $D = Z(C)$. Clearly C is a characteristic subgroup of P.

(a) Since C/D is elementary abelian, first $C/Z(C)$ is elementary and then $[C, C] \subseteqq D \subseteqq Z(C)$. Hence $cl\ C \leqq 2$.

(b) Since $C/D \subseteqq Z(P/D)$, $[P, C] \subseteqq D = Z(C)$.

(c) Suppose that $Q = C_P(C) \nsubseteqq C$. Since $Q \cap C = Z(C) = D$ we have $Q/D \subset P/D$ and $Q/D \cap C/D = 1$. Of course $Q \subseteqq C_P(C) \subseteqq C_P(D)$. If $Q \neq D$, then Q/D intersects $\Omega_1(Z(P/D)) \cap C_P(D)/D$ non-trivially.

This contradiction completes the proof. //

1. BAER'S THEOREM

The Theorem 1.1 is required in the study of p-stable groups and a proof due to Suzuki is given in [12]. Of course, it follows immediately from the result of Baer [15] p. 298, this proof being given below as the first proof. Two other proofs of this result are given, both of which are interesting and brief.

Theorem 1.1 (R. Baer). _Let_ K _be a conjugacy class of_ p-_elements in a finite group_ G. _If_ $\langle x, y\rangle$ _is a_ p-_group for all_ $x, y \in K$, _then_ $K \subseteqq O_p(G)$.

First Proof. Since $\langle x, y\rangle$ is a p-group for all $x, y \in K$, for all $g \in G$, $[x, g] = x^{-1}x^g$ and x are elements of the finite p-group $\langle x, x^g\rangle$. Hence

$$[g, x, x, \ldots, x] = 1 \text{ after a while.}$$

Thus x is a right Engel element and by Theorem III, 6.15 [15], $x \in F(G)$. Hence $K \subseteqq O_p(G)$. //

Second Proof (J. H. Walters). Let G be a minimal counter example to the Theorem. Let $M_1, M_2, \ldots, M_t$ be all the maximal subgroups of G containing a fixed element $x \in K$.

Clearly $O_p(G) = 1$ since G is a minimal counter example.

If $t = 1$, then for all $y \in K$, $\langle x, y\rangle \subseteqq M_1$, since $\langle x, y\rangle$ is a p-subgroup of G and so is certainly a proper subgroup of G containing x. Thus $K \subseteqq M_1$. Let $L = \langle K\rangle$. Then $L \subseteqq M_1 \subset G$ and $K \subseteqq O_p(L)$ by induction. Since $L \trianglelefteqq G$, we have a contradiction.

Thus we have $t > 1$. Among all i, j with $i \neq j$ choose $D = M_i \cap M_j$ such that $|D|_p$ is maximal. Let P be a Sylow p-subgroup of D containing x.

We show that there is no loss of generality in assuming that P is a Sylow p-subgroup of both M_i and M_j. For suppose that $P \subset P_i$, a Sylow p-subgroup of M_i. Then $N_G(P) \cap P_i \supset P$. Let M_k be a maximal subgroup of G containing $N_G(P) \subset G$. Then $M_k \cap M_i \supseteqq N_G(P) \cap P_i \supset P$ and so $k = i$ by the choice of i, j. Also $N_G(P) \subseteqq M_i$ and so P is a Sylow p-subgroup of M_j. Now choose $n \in N_G(P) \cap P_i - P$. Then clearly $n \notin M_j$ and so $M_j^n \neq M_j$. Otherwise $M_j \trianglelefteqq G$ and by induction $x \in K \cap M_j \subseteqq O_p(M_j) \subseteqq O_p(G) = 1$, a contradiction. Now $M_j^n \supseteqq P^n = P$ contains x and so $M_j^n = M_l$ for some l. Take $M_j \cap M_l$ as our required intersection. Note that P is a Sylow p-subgroup of both M_j and M_l.

We derive a contradiction easily now. By induction

$K \cap M_i \subseteq O_p(M_i) \subseteq P \subseteq M_j$.

Hence $K \cap M_i \subseteq K \cap M_j$.

Similarly $K \cap M_j \subseteq K \cap M_i$.

Thus $M_j = N_G(\langle K \cap M_j\rangle) = N_G(\langle K \cap M_i\rangle) = M_i$, a final contradiction. //

Third Proof (J. Alperin and R. Lyons). [1] Again let G be a minimal counter example. Let P be a Sylow p-subgroup of G. If $\langle K\rangle$ is a p-subgroup, then $K \subseteq O_p(G)$ since $K \trianglelefteq G$. Thus $\langle K\rangle$ is not a p-subgroup and so $K \nsubseteq P$. Let $y \in K - P$ and let Q be a Sylow p-subgroup of G containing y. Then of course $K \cap P \neq K \cap Q$.

Among all Sylow p-subgroups P, Q of G such that $K \cap P \neq K \cap Q$ choose P, Q so that $|K \cap P \cap Q|$ is maximal. Since $P^x = Q$ for some $x \in G$, $(K \cap P)^x = K \cap Q$ and so $K \cap P \nsubseteq Q$, $K \cap Q \nsubseteq P$. Let $D = \langle K \cap P \cap Q\rangle$. Suppose $D = P_0 \subset P_1 \subset \ldots \subset P_n = P$ where $[P_{i+1} : P_i] = p$.

Clearly $K \cap P \nsubseteq D$.

Suppose i is the smallest positive integer such that $K \cap P_i \nsubseteq K \cap D$. Let $x \in (K \cap P_i) - D$. Since $P_{i-1} \triangleleft P_i$, x normalizes P_{i-1} and so x normalizes $\langle K \cap P_{i-1}\rangle = D$. Choose $y \in (K \cap Q) - P$ similarly such that y normalizes D.

Then $\langle x, y\rangle$ is a p-group by hypothesis and so $\langle x, y, D\rangle$ is a p-group also. Let R be a Sylow p-subgroup of G containing $\langle x, y\rangle D$.

Then $\langle x, D\rangle \subseteq R \cap P$ implies that $R = P$ while $\langle y, D\rangle \subseteq R \cap Q$ implies that $R = Q$. This is a contradiction. //

2. A THEOREM OF BLACKBURN

This theorem duplicates some of the results of [12] - but its proof is so beautiful that it should be included here. The following lemma is of crucial importance for many of the results to come.

Lemma 2.1 (J. Thompson). Let a be a p'-automorphism of a p-group G. Suppose that X is a p-group of automorphisms of G and $[a, X] = [a, C_G(X)] = 1$. Then $a = 1$.

Proof. Let $N \subseteq G$ be X-invariant such that $[a, N] \neq 1$, but $[a, K] = 1$ for all X-invariant proper subgroups K of N. Then apply the

Three Subgroups Lemma. We have

$[N, X, a] = 1$ because $[N, X] \subset N$

and is X invariant.

$[X, a, N] = 1.$

Thus $[a, N, X] = 1$, $[N, a] \subseteq C_G(X)$, $[N, a, a] = 1$. By 0.2, $[N, a] = 1$. This completes the proof. //

Lemma 2.2. Let a be a π'-automorphism of a π-group G and suppose $X \lhd \lhd G$ is such that $[a, X] = [a, C_G(X)] = 1$. Then $a = 1$.

Proof. Let $X \lhd X_1 \lhd \ldots \lhd X_n = G$ and choose i such that $[a, X_{i+1}] \neq 1$, $[a, X_i] = 1$. Let $N = N_G(X_i)$. Since $X_{i+1} \subseteq N$, $[a, N] \neq 1$. But

$[X_i, N, a] = 1$
$[X_i, a, N] = 1.$

Hence $[N, a, X_i] = 1$, $[N, a] \subset C_G(X_i) \subseteq C_G(X)$. Thus $[N, a, a] = 1$. Lemma 0.2 implies that $[N, a] = 1$. //

Lemma 2.3 (N. Blackburn) [6]. Let a be a p'-automorphism of a p-group P. Let E be an abelian subgroup of P, maximal of exponent p^n, where $n \geq 2$ if P is a non-abelian 2-group and no restriction is placed on n otherwise. If $[a, E] = 1$, then $a = 1$.

Proof. Let P be a minimal counter example.

If $C = C_P(E) \subset P$, then $[a, C] = 1$ by induction. By 2.2, $a = 1$. Thus $E \subseteq Z(P)$.

Also, since $E \neq P$ trivially, $\Phi(P)E \subset P$. Thus a centralizes $\Phi(P)$ by induction. If $C(\Phi(P)) \subset P$, again we have $[a, \Phi(P)] = [a, C_P(\Phi(P))] = 1$ since $E \subset C(\Phi(P))$. By 2.2, $a = 1$ again. Thus P has class at most 2 and $\Phi(P) \subseteq Z(P)$.

Choose $x \in P$ and consider $[x, a]^{p^n}$.

$$[x, a]^{p^n} = (x^{-1}x^a)^{p^n} = x^{-p^n}(x^a)^{p^n}[x^a, x^{-1}]^{p^n(p^n-1)/2}.$$

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If P is abelian then of course $[x^a, x^{-1}] = 1$. On the other hand, if P is non-abelian, then

$$[x^a, x^{-1}]^{p^n(p^n-1)/2} = [x^a, x^{-p}]^{p^{n-1}(p^n-1)/2}$$

since if $p = 2$, $n \geqq 2$. But $x^{-p} \in \Phi(P) \subseteqq Z(P)$ for all $x \in P$. Thus in every case we have

$$[x, a]^{p^n} = x^{-p^n}(x^a)^{p^n} = x^{-p^n}(x^{p^n})^a.$$

Since $[a, \Phi(P)] = 1$ and $x^{p^n} \in \Phi(P)$, we see that $[x, a]^{p^n} = 1$. By the maximality of E, $[x, a] \in E$, for all $x \in P$. Thus $[P, a] \subseteqq E$ and $[P, a, a] = 1$. By 0. 2, $[P, a] = 1$ and so $a = 1$. //

Theorem 2. 4. Let P be a p-group. If a is a p'-automorphism of P which centralizes $\Omega_1(P)$ then $a = 1$ unless P is a non-abelian 2-group. If $[a, \Omega_2(P)] = 1$, then $a = 1$ without restriction. //

3. A THEOREM OF BENDER

Theorem 3. 1 [2]. Let G be a p-constrained group. If $p = 2$ assume that the Sylow 2-subgroups of G have class $\leqq 2$. Let E be an abelian p-subgroup of G which contains every p-element of its centralizer. Then every E-invariant p'-subgroup H of G lies in $O_{p'}(G)$.

Remark 1. If E is a self centralizing normal subgroup of a Sylow p-subgroup of G, then E contains every p-element of its centralizer in G. For let $E \in \mathcal{SCN}(P)$ where P is a Sylow p-subgroup of G. Suppose that $D \supseteqq E$ is a Sylow p-subgroup of $C_G(E)$. Consider $N_G(E)$. Suppose that Q is a Sylow p-subgroup of $N_G(E)$ containing D. Since $P \subseteqq N_G(E)$, there exists $n \in N_G(E)$ such that $Q^n = P$. Then $D^n \subseteqq P \cap C_G(E) = E$. Hence $D = E$ is a Sylow p-subgroup of $C_G(E)$. By Burnside's Theorem, $C_G(E) = E \times O_{p'}(C_G(E))$.

Remark 2. Theorem 3. 1 cannot hold without restriction if $p = 2$, even if G is solvable. Consider for example $G = GL(2, 3)$, $E \subseteqq G$ a fours-group. Then $O_{2'}(G) = 1$, $C_G(E) = E$, but there is a subgroup of order 3 which is normalized by E. Note that a Sylow 2-subgroup of $GL(2, 3)$ has class 3.

Proof. Let G be a minimal counter example. The proof proceeds by a series of steps.

1. $O_{p'}(G) = 1$.

Otherwise, let $\overline{G} = G/O_{p'}(G)$. Since $C_{\overline{G}}(\overline{E}) = \overline{C_G(E)}$, by 0.3, we have $\overline{H} \subseteqq O_{p'}(\overline{G}) = 1$. Thus $H \subseteqq O_{p'}(G)$.

Let $R = O_p(G)$ and let $Q \neq 1$ be a minimal E-invariant p'-subgroup of G. If $RQE \subset G$, then $Q \subseteqq O_{p'}(RQE)$ by induction. Hence $[R, Q] \subseteqq O_{p'}(RQE) \cap R = 1$. Since $C_G(R) \subseteqq R$ by p-constraint we have

2. $G = RQE$.

Let S be a QE-invariant subgroup of G minimal with respect to $[Q, S] \neq 1$. Then S is a special p-group. The argument which verifies this is standard. See for example [12].

If S is abelian, then $S = C_S(Q) \oplus [Q, S]$ by 0.5. But $[Q, S]$ is an E-invariant p-subgroup and so $C_P(E) \cap [Q, S] \neq 1$. Thus $E \cap [Q, S] \neq 1$ by our hypothesis on E. On the other hand $[E \cap [Q, S], Q] \subseteqq Q \cap S = 1$. This contradicts $C_S(Q) \cap [Q, S] = 1$.

If S is non-abelian and p is odd, we use a remarkable idea of Bender, or perhaps of Baer. First by 2.4, since $[Q, S] \neq 1$, and S is minimal, $S = \Omega_1(S)$ has exponent p. Let T be a new group defined as follows: T = S qua set.

Every element $x \in S$ has a unique square root $x^{\frac{1}{2}} \in S$, since p is odd. Define a binary operation o on T as follows $x \circ y = x^{\frac{1}{2}} y x^{\frac{1}{2}}$.

It is routine to check that T is an elementary abelian group. Also QE acts as a group of automorphisms of T. Since S = T as sets, the fixed points of both Q and E on T are unchanged. But we have already reached a contradiction when S is abelian. This same argument can be applied to TQE.

If $p = 2$ and S is non-abelian, first $[E, S] \subseteqq Z(S)$ since SE has class $\leqq 2$. Thus

$$[S, E, Q] \subseteqq [Z(S), Q] = 1$$
$$[Q, S, E] \subseteqq [S, E] \subseteqq Z(S).$$

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Hence

$$[E, Q, S] \subseteq Z(S).$$

If $[E, Q] \neq 1$, then $[E, Q] \subseteq Q$ stabilizes the chain $S \supseteq Z(S) \supseteq 1$. Hence $[E, Q] \subseteq C_G(S)$ by 0.1. Since $[E, Q]$ is an E-invariant subgroup of Q, minimality of Q ensures that $Q = [E, Q] \subseteq C_G(S)$. This is a contradiction.

Thus we may assume that $[E, Q] = 1$. Since $[S, E] \subset S$ is then Q-invariant, minimality of S ensures that $[S, E, Q] = 1$. Also $[E, Q, S] = 1$. It follows that $[Q, S, E] = [S, E] = 1$. Our assumptions on E now give $S \subseteq E$ and $[S, Q] \subseteq Q \cap S = 1$. This contradiction completes the proof. //

Lemma 3.2. *Suppose P is a p-subgroup of a p-constrained group G. Then* $O_{p'}(N_G(P)) \subseteq O_{p'}(G)$.

Proof. Since G is p-constrained, it follows from 0.3 that $G/O_{p'}(G)$ is p-constrained. Let $\overline{G} = G/O_{p'}(G)$ etc. By induction we have $O_{p'}(N_{\overline{G}}(\overline{P})) \subseteq O_{p'}(\overline{G}) = 1$. But clearly $O_{p'}(N_{\overline{G}}(\overline{P})) = O_{p'}(C_{\overline{G}}(\overline{P})) = \overline{O_{p'}(C_G(P))}$ by 0.3. Thus $O_{p'}(N_{\overline{G}}(\overline{P})) = \overline{O_{p'}(C_G(P))}$. Since $O_{p'}(N_G(P)) \subseteq C_G(P)$, it follows that $O_{p'}(N_G(P)) \subseteq O_{p'}(G)$ in this case. Hence we may assume that $O_{p'}(G) = 1$. Let $M = O_p(G)$, $Q = O_{p'}(N_G(P))$. Since $[Q, P] = 1$ and $[Q, C_M(P)] \subset M \cap Q = 1$ we have $[Q, M] = 1$ by 2.1. Hence $Q = 1$ because $C_G(M) \subseteq M$ by p-constraint.

Remark. The reader should refer to Lemma 12.5, 12.6 due to Bender for a far reaching generalization of this result.

Lemma 3.3. *If G is a p-solvable group of odd order and P is a Sylow p-subgroup of G such that* $r(P) \leq 2$, *then G has p-length* 1.

Proof. Let G be a minimal counter example. Clearly $O_{p'}(G) = 1$. Let $R = O_p(G)$. Since G is p-constrained, $C_G(R) \subseteq R$. Let C be a Thompson critical subgroup of R and let $D = \Omega_1(C)$. Since $|G|$ is odd and C has a class ≤ 2, D has exponent p.

Since $r(P) \leq 2$, $r(D) \leq 2$. If $|Z(D)| \geq p^2$, then $D = Z(D)$ and if

$|Z(D)| = p$ then any subgroup of type (p, p) containing $Z(D)$ has centralizer of index $\leqq p$. It follows that $|D| \leqq p^3$. Also $|D| = p^3$ only if D is non-abelian of exponent p. Let $\overline{D} = D/\Phi(D)$. Then $C_G(\overline{D})$ is still a normal p-subgroup of G and so $C_G(\overline{D}) \subseteqq R$. But $G/C_G(\overline{D}) \subseteqq GL(2, p)$. But any odd order subgroup of $GL(2, p)$ has a normal Sylow p-subgroup.

Thus $G = O_{p,p'}(G \bmod C_G(\overline{D}))$ and so $G = O_{p,p'}(G)$. //

Lemma 3.4. If G is a solvable group of odd order and P is a Sylow p-subgroup of G such that P' is cyclic, then G has p-length 1.

Proof. Let G be a minimal counter example. First $O_{p'}(G) = 1$ clearly. Let $R = O_p(G)$. Then $C_G(R) \subseteqq R$ since G is solvable.

If $\Phi(R) \neq 1$, let $\overline{G} = G/\Phi(R)$. Since $\overline{P}'$ is cyclic, $\overline{G}$ has p-length 1. Let $Q\Phi(R) = O_{p'}(G \bmod \Phi(R))$ where Q is a p'-group. Then $[Q, R] \subseteqq \Phi(R)$ and so Q centralizes R modulo $\Phi(R)$. Thus $[Q, R] = 1$. Hence $O_{p'}(G \bmod \Phi(R)) = 1$ and G has p-length 1.

Thus we have that R is an elementary abelian p-group. Let $x \in P$. Then $[x, R] \subseteqq P' \cap R$, a cyclic subgroup of order p. Thus

$$[x, R] \subseteqq Z(P) \text{ and } [R, x, x] = 1.$$

Hence x acts on R with quadratic minimum polynomial. For a discussion of this see the Appendix p. 80. By the famous Theorem B of Hall and Higman $[x, R] = 1$. Thus $P = R$ and the Lemma is proved. //

4. THE TRANSITIVITY THEOREM

Included here is a proof of a rather unsatisfactory form of the Thompson Transitivity Theorem. This is proved completely in [12]. The proof given here is shorter but the Theorem is less general. More precisely, for the case of odd primes, the Theorem is more general; but for $p = 2$ it deals only with groups, whose Sylow 2-subgroups are of class at most 2. I do not know of any slick way to prove the general result. The final difficulty arises from the fact that a subgroup can be self centralizing and normal in one Sylow p-subgroup of a group but contained in another Sylow p-subgroup non-normally. The Theorem 3.1 is