Introduction

The purpose of these notes is to give a geometrical treatment of generalised homology and cohomology theories. The central idea is that of a 'mock bundle', which is the geometric cocycle of a general cobordism theory, and the main new result is that any homology theory is a generalised bordism theory. Thus every theory has both cycles and cocycles; the cycles are manifolds, with a pattern of singularities depending on the theory, and the cocycles are mock bundles with the same 'manifolds' as fibres.

The geometric treatment, which we give in detail for the case of pl bordism and cobordism, has many good features. Mock bundles are easy to set up and to see as a cohomology theory. Duality theorems are transparent (the Poincaré duality map is the identity on representatives). Thom isomorphism and the cohomology transfer are obvious geometrically while cup product is just 'Whitney sum' on the bundle level and cap product is the induced bundle glued up. Transversality is built into the theory - the geometric interpretations of cup and cap products are extensions of those familiar in classical homology. Coefficients have a beautiful geometrical interpretation and the universal coefficient sequence is absorbed into the more general 'killing' exact sequence. Equivariant cohomology is easy to set up and operations are defined in a general setting. Finally there is the new concept of a generalised cohomology with a sheaf of coefficients (which unfortunately does not have all the nicest properties).

The material is organised as follows. In Chapter I the transition from functor on cell complexes to homotopy functor on polyhedra is axiomatised, the mock bundles of Chapter II being the principal example. In Chapter II, the simplest case of mock bundles, corresponding to pl cobordism, is treated, but the definitions and proofs all generalise to the more complicated setting of later chapters. In Chapter III is the geometric treatment of coefficients, where again only the simplest case,

pl bordism, is treated. A geometric proof of functoriality for coefficients is given in this case. Chapter IV extends the previous work to a generalised bordism theory and includes the 'killing' process and a discussion of functoriality for coefficients in general (similar results to Hilton's treatment being obtained). In Chapter V we extend to the equivariant case and discuss the $\mathbf{Z}_2$ operations on pl cobordism in detail, linking with work of tom Dieck and Quillen. Chapter VI discusses sheaves, which work nicely in the cases when coefficients are functorial (for 'good' theories or for 2-torsion free abelian groups) and finally in Chapter VII we prove that a general theory is geometric. The principal result is that a theory has cycles unique up to the equivalence generated by 'resolution of singularities'. The result is proved by extending transversality to the category of CW complexes, which can now be regarded as geometrical objects as well as homotopy objects. Any CW spectrum can then be seen as the Thom spectrum of a suitable bordism theory. The intrinsic geometry of CW complexes, which has strong connections with stratified sets and the later work of Thom, is touched on only lightly in these notes, and we intend to develop these ideas further in a paper.

Each chapter is self-contained and carries its own references and it is not necessary to read them in the given order. The main pattern of dependence is illustrated below.

$$\begin{tikzcd}
\mathrm{I} \arrow[d] & & \\
\mathrm{II} \arrow[d] \arrow[dr] \arrow[drr] & & \mathrm{III} \arrow[d] \\
\mathrm{IV} \arrow[d] & \mathrm{V} & \mathrm{VI} \\
\mathrm{VII} & &
\end{tikzcd}$$

The germs of many of the ideas contained in the present notes come from ideas of Dennis Sullivan, who is himself a tireless campaigner for the geometric approach in homology theory, and we would like to dedicate this work to him.

NOTE ON INDEXING CONVENTIONS

Throughout this set of notes we will use the opposite of the usual convention for indexing cohomology groups. This fits with our geometric description of cocycles as mock bundles - the dimension of the class then being the same as the fibre dimension of the bundle. It also means that coboundaries reduce dimension (like boundaries), that both cup and cap products add dimensions and that, for a generalised theory, $h^n(\mathrm{pt.}) \cong h_n(\mathrm{pt.})$. However the convention has the disadvantage that ordinary cohomology appears only in negative dimensions. If the reader wishes to convert our convention to the usual one he has merely to change the sign of the index of all cohomology classes.

I·Homotopy functors

The main purpose of this chapter is to axiomatise the passage from functors defined on pl cell complexes to homotopy functors defined on polyhedra. Principal examples are simplicial homology and mock bundles (see Chapter II).

Our main result, 3.2, states that the homotopy category is isomorphic to the category of fractions of pl cell complexes defined by formally inverting expansions. Thus to define a homotopy functor, it is only necessary to check that its value on an expansion is an isomorphism. Analogous results for categories of simplicial complexes have been proved by Siebenmann, [3].

In §4, similar results are proved for Δ-sets. This gives an alternative approach to the homotopy theory of Δ-sets (compare [2]).

In §6 and §7, we axiomatise the construction of homotopy functors and cohomology theories. Here we are motivated by the coming application to mock bundles in Chapter II, where the point of studying cell complexes, rather than simplicial complexes, becomes plain as the Thom isomorphism and duality theorems fall out.

The idea of using categories of fractions comes (to us) from [1] where results, similar to those contained in §4 here, are proved.

Throughout the notes we use basic pl concepts; for definitions and elementary results see [4, 6 or 8].

1. DEFINITIONS

Ball complexes

Let K be a finite collection of pl balls in some R^n, and write $|K| = \cup\{\sigma : \sigma \in K\}$. Then K is a <u>ball complex</u> if

(1) $|K|$ is the disjoint union of the interiors $\mathring{\sigma}$ of the $\sigma \in K$, and

(2) $\sigma \in K$ implies the boundary $\dot{\sigma}$ is a union of balls of K.

It then follows that

(3) if σ, $\tau \in K$, then $\sigma \cap \tau$ is a union of balls of K.

Notice that we do not assume $\sigma \cap \tau \in K$.

A subset $L \subset K$ is a subcomplex if L is itself a ball complex, and we write (K, L) for such a pair. If (K_0, L_0) is another pair, and $K_0 \subset K$, $L_0 \subset L$ are subcomplexes, then there is the inclusion $(K_0, L_0) \subset (K, L)$. An isomorphism $f : (K, L) \to (K_1, L_1)$ is a $p\ell$ homeomorphism $f : |K| \to |K_1|$ such that $f|L| = |L_1|$, and $\sigma \in K$ implies $f(\sigma) \in K_1$. In the case where K and K_1 are simplicial complexes, there are simplicial maps $f : (K, L) \to (K_1, L_1)$. The product $K \times L$ of ball complexes K, L is defined by $K \times L = \{\sigma \times \tau \mid \sigma \in K,\ \tau \in L\}$.

The categories Bi and Bs

Now define the category Bi to have for objects pairs (K, L) and morphisms generated by isomorphisms and inclusions (i.e., a general morphism is an isomorphism onto a subpair). The category Bs has the same objects but the generating set for the morphisms is enlarged to include simplicial maps between pairs of simplicial complexes.

Subdivisions

If L', L are ball complexes with each ball of L' contained in some ball of L and $|L'| = |L|$, we say L' subdivides L, and write $L' \lhd L$. The categories Bi and Bs enjoy a technical advantage over categories of simplicial complexes; namely, if $L \subset K$ and $L' \lhd L$, then there is a complex $L' \cup K = \{\sigma : \sigma \in L'$ or $K - L\}$.

Collapsing

We assume familiarity with the notion of collapsing, as in [6], for example. Suppose $(K_0, L_0) \subset (K, L)$, where $L_0 = L \cap K_0$; then we have a collapse $(K, L) \searrow (K_0, L_0)$ if $K \searrow K_0$ and any elementary collapse in the sequence from a ball in L is across a ball in L (so that in particular, $L \searrow L_0$). We call the inclusion $(K_0, L_0) \subset (K, L)$ an expansion. The composition of an expansion with an isomorphism is still called an expansion.

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Cambridge University Press
978-0-521-20940-3 - A Geometric Approach to Homology Theory
S. Buoncristiano, C. P. Rourke, and B. J. Sanderson
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2. SUBDIVISION IN THE CATEGORY OF FRACTIONS

Let $B = Bi$ or Bs, and let Σ denote the set of expansions. The category of fractions $B[\Sigma^{-1}]$ is formed by formally inverting expansions. Thus the objects are the same. New morphisms e^{-1}, $e \in \Sigma$, are introduced, and a morphism in the category of fractions is then an equivalence class of formal compositions $g_1 \circ g_2 \circ \dots \circ g_n$, where $g_i \in B$ or $g_i = e_i^{-1}$ for some $e_i \in \Sigma$. The equivalence relation is generated by the following operations:

(i) replace h by $f \circ g$ if $h = fg$ and $f, g \in B$;

(ii) introduce $e \circ e^{-1}$ or $e^{-1} \circ e$, $e \in \Sigma$;

(iii) replace $(e_2 e_1)^{-1}$ by $e_1^{-1} \circ e_2^{-1}$.

In fact operation (iii) is a consequence of operations (i) and (ii). Denote the equivalence class of a formal composition by $\{g_0 \circ g_1 \circ \dots \circ g_n\}$.

The category of fractions is characterised by a universal mapping property; namely, given any functor $F : B \to C$ such that $F(e)$ is an isomorphism for each $e \in \Sigma$, then there exists a unique functor F' so that

$$\begin{tikzcd} B \arrow[rr, "F"] \arrow[dr, "p"'] & & C \\ & B[\Sigma^{-1}] \arrow[ur, "F'"'] & \end{tikzcd}$$

commutes, where p is the natural map.

For simplicity, in the rest of the paper we will ignore pairs (K, L) with $L \neq \emptyset$ when the general case can be obtained by making minor adjustments. We first observe that any morphism in $Bi[\Sigma^{-1}]$ may in fact be written $\{e^{-1} \circ f\}$ by repeated use of the following lemma.

Lemma 2.1. *Let* $e : J \to K$ *be an expansion and* $f : J \to L$ *a morphism in* Bi. *Then there is an expansion* e_0 *and morphism* f_0 *so that*

$$\begin{tikzcd} & J \arrow[dl, "e"'] \arrow[dr, "f"] & \\ K \arrow[dr, "f_0"'] & & L \arrow[dl, "e_0"] \\ & J_0 & \end{tikzcd}$$

commutes.

Proof. Define $J_0 = K \cup_J L$, where J is regarded as a subcomplex of both by e and f. Then let e_0 and f_0 be the obvious inclusions. That e_0 is an expansion follows by echoing the collapse $K \searrow J$.

Remark. The lemma fails for Bs. For instance, take $K = I \times I$, $J = \{0\} \times I$, and $L = \{0\}$. Then f_0 must be degenerate on eJ and hence be degenerate on the 2-cell in K.

Now suppose $L' \lhd L$; then there are inclusions $i : L \to L \times I \cup L' \times \{1\}$ and $e : L' \to L \times I \cup L' \times \{1\}$. Then e is an expansion, and so we have a morphism $\{e^{-1} \circ i\} : L \to L'$ in $B[\Sigma^{-1}]$, called *subdivision* and denoted $\rhd(L, L')$.

Lemma 2.2. $\rhd$ *is functional; that is,*

(i) $\rhd(L, L) =$ *identity,*

(ii) *if* $L'' \lhd L' \lhd L$, *then*

$$\rhd(L', L'') \circ \rhd(L, L') = \rhd(L, L'').$$

Proof. For part (i), we must show that if $i_0, i_1 : K \to K \times I$ are the inclusions, then $\{i_0\} = \{i_1\}$. This is proved by simple collapsing arguments. First, consider $K \times I$ subdivided to $(K \times I)'$ by placing K at $K \times \{\frac{1}{2}\}$. There are then inclusions i'_0, i'_1, $i_{\frac{1}{2}}$ of K in $(K \times I)'$ and a reflection $r : (K \times I)' \to (K \times I)'$ about the half-way level. Now $ri_{\frac{1}{2}} = i_{\frac{1}{2}}$, and $\{i_{\frac{1}{2}}\}$ is an isomorphism since $(K \times I)' \searrow K \times \{\frac{1}{2}\}$. It follows that $\{r\} =$ identity. Since $ri'_0 = i'_1$, we have $\{i'_0\} = \{i'_1\}$. The result then follows by considering $K \times \Delta^2$. This argument is essentially due to Siebenmann [3; p. 480].

For part (ii), let Δ^2 be a 2-simplex with vertices v_0, v_1, v_2 and opposite faces $\Delta^1_0, \Delta^1_1, \Delta^1_2$. Let $(L \times \Delta^2)' = L \times \Delta^2 \cup L' \times \Delta^1_0 \cup L'' \times \{v_2\}$. Then $(L \times \Delta^2)' \searrow L' \times \Delta^1_0 \cup L \times \Delta^1_1 \cup L'' \times \{v_2\} \searrow L'' \times \{v_2\}$. The proof is completed by a diagram chase. See Fig. 1.

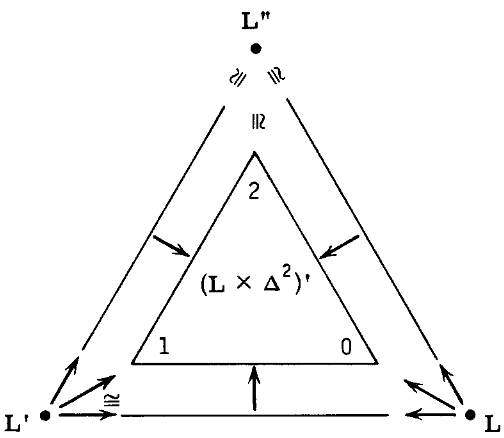

Fig. 1

Lemma 2.3. <u>Suppose given</u> $L' \lhd L$; <u>then there exists</u> $L'' \lhd L'$ <u>such that</u> $(L \times I) \cup L'' \times \{1\} \searrow L \times \{0\}$.

Proof. Let $\sigma_1, \ldots, \sigma_n$ be the balls of L listed in order of decreasing dimensions. We subdivide L in n steps. After step r, the interiors of $\sigma_1, \ldots, \sigma_r$ are not touched again. Suppose r-steps completed, and let $\sigma = \sigma_{r+1}$. Then σ has been subdivided to σ', say. Let $\tau \in \sigma'$ be a cell with $\dim \tau = \dim \sigma$, and $\tau \cap \dot{\sigma} = \emptyset$. If no such τ exists, perform a preliminary subdivision of σ. Now assume there is such a τ. Then $\sigma_c = \sigma$ - interior (τ) is a collar on $\dot{\sigma}$ in σ, and $\sigma_c \subset \sigma'$. Subdivide σ_c to σ''_c so that a collar projection $\sigma''_c \to \dot{\sigma}''$ is simplicial. $\sigma'' = \tau \cup \sigma''_c$ is the required subdivision of σ. Note the cylindrical collapse $\sigma''_c \searrow \dot{\sigma}''$.

The resulting complex L'' clearly has the desired property since

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978-0-521-20940-3 - A Geometric Approach to Homology Theory
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$$(\sigma \times I) \cup \sigma'' \times \{1\} \searrow (\dot{\sigma} \times I) \cup \sigma \times \{0\} \cup \sigma''_{c} \times \{1\} \searrow (\dot{\sigma} \times I) \cup \sigma \times \{0\},$$

where the first collapse is elementary, and the second is cylindrical.

Corollary 2.4. <u>Suppose</u> $L' \lhd L$; <u>then</u> $\rhd (L, L') : L \to L'$ <u>is an isomorphism.</u>

Proof. It follows from 2.3 that there is $L'' \lhd L'$ and $L''' \lhd L''$, so that $\rhd (L, L'')$ and $\rhd (L', L''')$ are isomorphisms. The result now follows from 2.2.

3. ISOMORPHISM WITH THE HOMOTOPY CATEGORY

Now let Bh denote the category with objects pairs of ball complexes, and morphisms homotopy classes of continuous maps. Then there are natural maps

$$\begin{tikzcd}[row sep=small] \mathrm{Bi} \arrow[dr] & \\ \cap & \mathrm{Bh}, \\ \mathrm{Bs} \arrow[ur] & \end{tikzcd}$$

and by the universal mapping property, we have a diagram

$$\begin{tikzcd} \mathrm{Bi}[\Sigma^{-1}] \arrow[dr, "\alpha"] \arrow[dd, "\beta"'] & \\ & \mathrm{Bh}\,. \\ \mathrm{Bs}[\Sigma^{-1}] \arrow[ur, "\gamma"'] & \end{tikzcd} \tag{3.1}$$

Theorem 3.2. <u>The maps in diagram (3.1) are isomorphisms of categories.</u>

Proof. Since all three categories have the same objects, it suffices to show that each of α, β, γ is an isomorphism on the set of morphisms from K to L for any K and L. We prove this in three steps:

A. α is surjective,

B. α is injective,

C. β is surjective.

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The result then follows by commutativity. We first observe that α and γ are compatible with subdivision; i. e.,

Remark 3. 3. Suppose $L' \lhd L$; then $\alpha(\rhd(L, L'))$ is the homotopy class of the identity map.

Step A. α is surjective.

Suppose $[f] : K \to L$ is a homotopy class; then by the pl approximation theorem,* there exist subdivisions $K' \lhd K$, $L' \lhd L$, and a simplicial map $f' : K' \to L'$ so that $[f'] = [f]$. Let $M(f')$ be the simplicial mapping cylinder of f'. There is then an inclusion $i : K' \to M(f')$, and an expansion $e : L' \to M(f')$. It follows from 3. 3 that

$$[f] = \alpha(\rhd (L, L')^{-1}\{e^{-1}\circ i\} \rhd (K, K')).$$

Step B. α is injective.

By 3. 1, it is sufficient to show that for any diagram

$$\begin{tikzcd}
 & J_0 & \\
K \arrow[ur, "f_0"] \arrow[dr, "f_1"'] & & L \arrow[ul, "e_0"'] \arrow[dl, "e_1"] \\
 & J_1 &
\end{tikzcd}$$

in which the e_i are expansions, and such that $\alpha\{e_0^{-1} \circ f_0\} = \alpha\{e_1^{-1} \circ f_1\}$, we have $\{e_0^{-1} \circ f_0\} = \{e_1^{-1} \circ f_1\}$. Now $\alpha\{e_0^{-1} \circ f_0\} = \alpha\{e_1^{-1} \circ f_1\}$ means that the diagram homotopy commutes, provided we regard e_0 and e_1 as homotopy equivalences.

Now let $J = J_0 \cup_L J_1$. Then by homotopy commutativity and the relative pl approximation theorem (see footnote), there is a simplicial map $f : (K \times I)' \to J'$ so that $f \big| \, |K| \times \{i\} = f_i$, $i = 0, 1$. Now consider the following diagram in which arrows marked e are expansions, and arrows marked $\rightarrowtriangle$ are compositions of subdivision followed by inclusion:

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* This is weaker than the usual simplicial approximation theorem. See [4] or [8] for a short proof.