Introduction

In these notes we are concerned with algebraic conditions on a linear operator from one Banach space into another that force the continuity of the linear operator. The main results are in the theory of Banach algebras, where the continuity of homomorphisms under suitable hypotheses is part of the standard theory (see Rickart [103], and Bonsall and Duncan [18]). The continuity of a multiplicative linear functional on a unital Banach algebra is the seed from which these results on the automatic continuity of homomorphisms grew, and is typical of the conditions on a linear operator that imply its continuity. Homomorphisms, derivations, and linear operators intertwining with a pair of continuous linear operators are the most important general classes of linear operators whose automatic continuity has been studied. These notes are an attempt to collect together and unify some of the results on the automatic continuity of homomorphisms and intertwining operators.

The most important results in these notes are in sections 4, 6, 8, 9, 10, and 12 of Chapters 2 and 3. The guiding problem behind Chapter 2 is to find necessary and sufficient conditions on a pair (T, R) of continuous linear operators on Banach spaces X, Y, respectively, so that each linear operator S from X into Y satisfying $ST = RS$ is continuous (Johnson [58]). The equivalent problem for homomorphisms is to find necessary and sufficient conditions on a pair of Banach algebras A and B so that each homomorphism from A into B (or onto B) is continuous (Rickart [103, §5]) (Chapter 3). Chapter 1 contains the general technical results on which Chapters 2 and 3 are built, and in Chapter 4 the continuity of positive linear functionals on a Banach $*$-algebra is discussed.

Throughout these notes all linear spaces will be over the complex field unless the space is explicitly stated to be over another field, and all linear operators will be complex linear except in Section 7 where we consider ring, i.e. rational linear, isomorphisms between semisimple Banach algebras. Many of the results do hold for real Banach spaces but we shall not consider them. Attention will be restricted to Banach spaces, and automatic continuity results for Fréchet spaces, and other topological linear spaces are not discussed. For example, the weak continuity of derivations on a von Neumann algebra (Kadison [70]), and the uniqueness of the Fréchet topology on the algebra $\mathbf{C}[[\chi]]$ of all formal power series in one indeterminate χ over the complex field $\mathbf{C}$ (Allan [1]) are omitted. Zorn's Lemma will be assumed throughout these notes and plays a crucial role in the existence of counter examples (Johnson [58, p. 88]). Axiomatic systems that imply the continuity of all linear operators between two infinite dimensional Banach spaces are beyond the scope of these notes (Wright [131]).

The reader will be assumed to know the basic theorems of functional analysis and elementary Banach algebra theory. In particular the following two results will be used frequently without reference: a finite dimensional normed linear space is complete; each linear operator from a finite dimensional normed linear space into a normed linear space is continuous. In certain sections deeper results from algebra and analysis are assumed. In Section 3 the properties of divisible, injective, and torsion modules over the principal ideal domain $\mathbf{C}[\chi]$ of all polynomials in an indeterminate χ over the complex field $\mathbf{C}$ are used in obtaining a discontinuous intertwining operator (Cartan and Eilenberg [22], Kaplansky [138]). The decomposition of a torsion module over $\mathbf{C}[\chi]$ is used in the proof of Theorem 4.1 (see Hartley and Hawkes [144]). The spectral theorem for normal operators on a Hilbert space is used in Section 5. The Wedderburn theory for finite dimensional semisimple algebras over $\mathbf{C}$ is required in Section 7 when we consider ring isomorphisms between semisimple Banach algebras (Jacobson [51]). In the same section, and in Section 8, we apply the single variable analytic functional calculus to an element in a Banach algebra (Bonsall and Duncan [18], or Rickart [103]). Elementary properties of field extensions, and the embedding of a domain in its field of fractions are used in Section 8 (Jacobson [50]). Section 12 requires elementary properties of C*-algebras (Dixmier [34]).

We shall now describe the contents of the various sections in more detail. The first section contains the basic properties of the separating space $\mathfrak{S}(S)$ of a linear operator S from a Banach space X into a Banach space Y, where

$$\mathfrak{S}(S) = \{y \in Y: \text{there is a sequence } (x_n) \text{ in } X \text{ with } x_n \to 0 \text{ and } Sx_n \to y\}.$$

The closed graph theorem implies that S is continuous if and only if $\mathfrak{S}(S) = \{0\}$. It is this equivalence that has made the separating space a useful technical device in automatic continuity problems. The most important result in this section is Lemma 1.6, which is used in Sections 4 and 11 (Johnson and Sinclair [69], Allan [1], Sinclair [118]). Lemma 1.6 shows that if (T_n) and (R_n) are sequences of continuous linear operators on Banach spaces X and Y, respectively, and if $ST_n = R_nS$ for all n, then there is an integer N such that

$$(R_1 \ldots R_n \mathfrak{S}(S))^- = (R_1 \ldots R_N \mathfrak{S}(S))^-$$

for all $n \geq N$.

In Section 2 we consider discontinuity points of an operator which leaves a large lattice of closed linear subspaces of a Banach space invariant. This idea seems to have been first used by Bade and Curtis [7], though not under this name, and has subsequently been exploited by many authors (for example Curtis [27], Gvozdková [48], Johnson [58], [63], [64], [66], Johnson and Sinclair [69], Ringrose [105], Sinclair [117], Stein [121], [122], Vrbová [128]). The conclusion when this method is used is that the discontinuity is concentrated in a subspace that is small in a technical sense associated with the lattice.

In Section 3 we prove that there exists a discontinuous linear operator S from a Banach space X into a Banach space Y satisfying $ST = RS$ under two additional hypotheses on T and R, where T and R are in $\mathcal{L}(X)$ and $\mathcal{L}(Y)$, respectively, and $\mathcal{L}(X)$ is the Banach algebra of continuous linear operators on the Banach space X (Johnson [58], Johnson and Sinclair [69], Sinclair [116]).

In Theorem 4.1 necessary and sufficient conditions are given on the pair (T, R), when R has countable spectrum, so that each linear operator S from X into Y satisfying ST = RS is continuous (Johnson and Sinclair [69], Sinclair [116]).

In Section 5 the operators R and T are assumed to be normal operators on Hilbert spaces and S intertwining with them is decomposed into continuous and highly discontinuous parts (Johnson [58]). The automatic continuity results that hold for other operators T and R with suitable spectral decompositions are not discussed in these notes (Johnson [58], Johnson and Sinclair [128], Vrbová [128]).

In Section 6 enough of the theory of irreducible modules over a Banach algebra is developed to prove the uniqueness of the complete norm topology of a semisimple Banach algebra (Corollary 6.13) (Johnson [59]). A full discussion of irreducible modules over a Banach algebra (and irreducible representations of a Banach algebra) may be found in Rickart [103] or Bonsall and Duncan [18]. Our proof of Theorem 6.9, on which the uniqueness of the complete norm topology of a semisimple Banach algebra depends, is no shorter than Johnson's original proof [59], but by basing it on Section 2 its relation to other automatic continuity proofs is emphasized. In Theorem 6.16 some properties of the spectrum of an element of the separating space of a homomorphism are given (Barnes [13]). From this we deduce the continuity of a homomorphism from a Banach algebra onto a dense subalgebra of a strongly semisimple unital Banach algebra (Rickart [101], Yood [132]).

In Section 7 we prove Kaplansky's Theorem [74] that decomposes a ring isomorphism between two semisimple Banach algebras into a linear part, a conjugate linear part, and a non real linear part on a finite dimensional ideal. This is proved using automatic continuity methods in a similar way to that in which the corresponding result for derivations was proved (Johnson and Sinclair [68]).

In Section 8 we briefly consider the relationship between discontinuous derivations from a Banach algebra A into Banach A-modules and discontinuous derivations from A. From a discontinuous derivation a discontinuous homomorphism may be constructed (Theorem 8.2). Dales's example of a discontinuous derivation from the disc algebra into a Banach module over it is the main result of this section [28]. The structure of the proof given here is slightly different from his but the idea is the same. The existence of a discontinuous homomorphism from the disc algebra into a suitable Banach algebra was first proved using Allan's theorem that embeds the algebra of all formal power series in one indeterminate into suitable Banach algebras [1], (see Johnson [66]). The proof of this excellent deep result is based on several complex variable theory, and was beyond the scope of these lectures.

Section 9 contains the main lemma, Theorem 9. 3, on which Sections 10 and 12 are based. The hypotheses of this theorem have been chosen to suit these two applications.

Section 10 is devoted to Bade and Curtis's theorem on the decomposition of a homomorphism from $\mathbf{C}(\Omega)$, where Ω is a compact Hausdorff space, into continuous and discontinuous parts (Theorem 10. 3) [7]. This is one of the most important results in automatic continuity, and the source of many ideas for subsequent research. A corollary (10. 4) of this theorem is that there is a discontinuous homomorphism from $\mathbf{C}(\Omega)$ into a Banach algebra if and only if there is a μ in Ω and a discontinuous homomorphism from $\mathbf{C}_0(\Omega\backslash\{\mu\})$ into a radical Banach algebra.

In Section 11 properties of a discontinuous homomorphism from $\mathbf{C}_0(\Psi)$ into a radical Banach algebra are studied, where Ψ is a locally compact Hausdorff space. The results of this section depend on Lemma 1. 6 and the observation that positive elements in $\mathbf{C}_0(\Psi)$ have positive roots (Sinclair [118]).

Section 12 contains some results on the continuity of homomorphisms and derivations from C*-algebras. In Corollary 12. 4 we prove that if a unital C*-algebra has no closed cofinite ideals (e. g. $\mathfrak{L}(H)$, where H is an infinite dimensional Hilbert space), then each homomorphism from it into a Banach algebra is continuous (Johnson [64]). In Corollary 12. 5 we show that a derivation from a C*-algebra into a Banach bimodule over it is continuous (Ringrose [105] and, see also, Johnson and Parrott [67]).

In Section 13 the standard results on the continuity of positive linear functionals on a Banach *-algebra are proved. The automatic continuity of positive linear functionals on other ordered Banach spaces is not considered (see Namioka [94], Peressini [98]). We shall also not consider certain other problems on the automatic continuity of linear functionals. For example, if a linear functional f on a C*-algebra A is continuous on all C*-subalgebras of A generated by single hermitian elements, is f continuous on A (Barnes [13], Barnes and Duncan [14], Ringrose [106])?

These notes are based on a course of postgraduate lectures given at the University of Edinburgh during the spring term 1974. Sections 7 and 13 were not given in the lectures, and Sections 3 and 8 were not covered in detail. I am indebted to those who participated for their suggestions, comments, and perseverance, and to F. F. Bonsall for encouraging me to give the lectures and write the notes. J. Cusack and N. P. Jewell read the manuscript, and their criticism and corrections have prevented many obscurities and errors. I am grateful for their advice. I should like to thank G. R. Allan, J. Cusack, H. G. Dales, B. E. Johnson, T. Lenegan, J. R. Ringrose, and many other friends for discussions, comments, letters, and preprints.

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1 · Technical results

In this chapter we develop some technical results needed in the subsequent chapters. In Section 1 we study the separating space $\mathfrak{S}(S)$ of a linear operator S from a Banach space X into a Banach space Y, where

$$\mathfrak{S}(S) = \{y \in Y: \text{ there is a sequence } (x_n) \text{ in } X \text{ with } x_n \to 0 \text{ and } Sx_n \to y\}.$$

The separating space is a useful tool in automatic continuity since S is continuous if and only if $\mathfrak{S}(S) = \{0\}$. It has been used by many authors to obtain the continuity of homomorphisms, derivations, module homomorphisms, and intertwining operators (for example [103], [68], [117], [69]). This tradition is followed in these notes. The proof of Lemma 1.6 illustrates the typical rolling hump argument of automatic continuity proofs.

The main result in Section 2 concerns the continuity behaviour of a linear operator with a large lattice of closed invariant subspaces, with properties akin to the open subsets of a compact Hausdorff space. This method of relating the discontinuity of the linear operator to a finite number of points in an associated topological space occurs in various forms in the following papers: [27], [48], [58], [63], [64], [66], [69], [105], [117], [121], [128]. Theorem 2.3 does not have as wide an application as we should wish but we are able to apply it later to study the continuity of a linear operator intertwining with a pair of normal operators, to prove the uniqueness of the complete norm topology on a semisimple Banach algebra, and to handle problems concerning additive operators.

1. The separating space

1.1. **Definition.** If S is a linear operator from a Banach space

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X into a Banach space Y, we let $\mathfrak{S}(S)$ or $\mathfrak{S}$ denote the set

$$\{y \in Y: \text{ there is a sequence } (x_n) \text{ in } X \text{ with } x_n \to 0 \text{ and } Sx_n \to y\},$$

and call it the separating space of S.

The first three lemmas contain the elementary properties of the separating space that we shall require in later chapters, and these lemmas will often be used without reference.

1.2. Lemma. Let S be a linear operator from a Banach space X into a Banach space Y. Then

(i) $\mathfrak{S}$ is a closed linear subspace of Y,

(ii) S is continuous if and only if $\mathfrak{S} = \{0\}$, and

(iii) if T and R are continuous linear operators on X and Y, respectively, and if $ST = RS$, then $R\mathfrak{S} \subseteq \mathfrak{S}$.

Proof. (i) The separating space is trivially a linear subspace of Y. Let (y_n) be a sequence in $\mathfrak{S}$ with $y_n \to y$ in Y. Choose a sequence (x_n) in X so that $\|x_n\| < 1/n$ and $\|Sx_n - y_n\| < 1/n$ for all n. Then $x_n \to 0$ and $Sx_n \to y$ as n tends to infinity. Hence $\mathfrak{S}$ is closed.

(ii) This is just the closed graph theorem in a different notation. If $\mathfrak{S} = \{0\}$, then S has a closed graph because $x_n \to x$ and $Sx_n \to y$ imply that $x_n - x \to 0$ and $S(x_n - x) \to y - Sx$ so that $y = Sx$.

(iii) If $x_n \to 0$ and $Sx_n \to y$, then $Tx_n \to 0$ and $STx_n = RSx_n \to Ry$.

1.3. Lemma. Let S be a linear operator from a Banach space X into a Banach space Y, and let R be a continuous linear operator from Y into a Banach space Z. Then

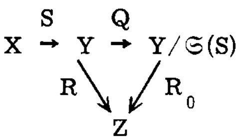
(i) RS is continuous if and only if $R\mathfrak{S}(S) = \{0\}$,

(ii) $(R\mathfrak{S}(S))^- = \mathfrak{S}(RS)$, and

(iii) there is a constant M (independent of R and Z) such that if RS is continuous then $\|RS\| \leq M\|R\|$.

Proof. (i) If RS is continuous, $x_n \to 0$, and $Sx_n \to y$, then $RSx_n \to Ry$ and $RSx_n \to 0$ so that $Ry = 0$.

Conversely suppose that $R\mathfrak{S}(S) = \{0\}$. The continuity of RS will follow from the commutativity of the diagram



once we have proved QS continuous; where Q is the natural quotient operator $Y \to Y/\mathfrak{S}(S) : y \mapsto y + \mathfrak{S}(S)$, and $R_0(y + \mathfrak{S}(S)) = Ry$. Let $x_n \to 0$ in X and $QSx_n \to y + \mathfrak{S}(S)$ in $Y/\mathfrak{S}(S)$. Then there is a sequence (y_n) in $\mathfrak{S}(S)$ such that $Sx_n - y - y_n \to 0$. We choose a sequence (w_n) in X so that $\|w_n\| < 1/n$ and $\|Sw_n - y_n\| < 1/n$. Then $x_n - w_n \to 0$ and $S(x_n - w_n) - y \to 0$ as $n \to \infty$ so that y is in $\mathfrak{S}(S)$. Thus $\mathfrak{S}(QS)$ is $\{0\}$ and QS is continuous.

(ii) We have $R\mathfrak{S}(S) \subseteq \mathfrak{S}(RS)$ because $x_n \to 0$ and $Sx_n \to y$ imply that $RSx_n \to Ry$. Since $\mathfrak{S}(RS)$ is closed, it follows that $(R\mathfrak{S}(S))^- \subseteq \mathfrak{S}(RS)$. Let $Q_0 : Z \to Z/(R\mathfrak{S}(S))^- : z \mapsto z + (R\mathfrak{S}(S))^-$. Then $Q_0R\mathfrak{S}(S)$ is null so that Q_0RS is continuous by (i), and thus $Q_0\mathfrak{S}(RS)$ is null also by (i). Therefore

$$\mathfrak{S}(RS) \subseteq (R\mathfrak{S}(S))^-.$$

(iii) Using the proof of (i) and $\|R\| = \|R_0\|$ we obtain

$$\|RS\| = \|R_0QS\| \le \|QS\| . \|R_0\| = \|QS\| . \|R\| .$$

Let $M = \|QS\|$, and the proof is complete.

From the above lemma it follows that $S^{-1}\mathfrak{S}(S)$ is closed because it is just Ker QS, where Q is defined as in the proof of (i).

1.4. Lemma. *Let X_0 and Y_0 be closed linear subspaces of Banach spaces X and Y, and let S be a linear operator from X into Y such that $SX_0 \subseteq Y_0$. Let $S_0 : X/X_0 \to Y/Y_0$ be defined by $S_0(x + X_0) = Sx + Y_0$. Then S_0 is continuous if and only if $Y_0 \supseteq \mathfrak{S}(S)$.*

Proof. If S_0 is continuous, $x_n \to 0$, and $Sx_n \to y$, then $S_0(x_n + X_0)$ tends to Y_0 and to $y + Y_0$ so that $\mathfrak{S}(S) \subseteq Y_0$. Conversely suppose that $\mathfrak{S}(S) \subseteq Y_0$. Let $Q : Y \to Y/Y_0 : y \mapsto y + Y_0$. Then QS is continuous and QS annihilates X_0 so that $S_0(x + X_0) = QS(x)$ and S_0 is continuous.

With the hypotheses of the above lemma we also have $\mathfrak{S}(S|X_0) \subseteq Y_0 \cap \mathfrak{S}(S)$, where $S|X_0$ is the restriction of S to X_0. This inclusion can be strict.

When R is a continuous linear operator, the above lemmas adequately describe the behaviour of the separating space of RS in terms of that of S. This raises the question of how does the separating space of ST behave for T a continuous linear operator from a Banach space into X. The general situation for ST is not as nice as that for RS. Clearly $\mathfrak{S}(ST) \subseteq \mathfrak{S}(S)$ but equality does not hold in general as one can see if S annihilates the range of T. The following result, which is a direct application of the open mapping theorem, is occasionally useful.

1.5. **Lemma.** *Let X, $Z_1, \ldots, Z_n$ be Banach spaces, and let $T_1, \ldots, T_n$ be continuous linear operators from $Z_1, \ldots, Z_n$ into X, respectively, such that $X = T_1Z_1 + \ldots + T_nZ_n$. Let S be a linear operator from X into a Banach space Y. Then $\mathfrak{S}(S) = (\mathfrak{S}(ST_1) + \ldots + \mathfrak{S}(ST_n))^-$.*

Proof. Suppose that $ST_1, \ldots, ST_n$ are continuous. Let $Z = Z_1 \oplus \ldots \oplus Z_n$ with norm $\|(z_1, \ldots, z_n)\| = \sum_1^n \|z_j\|$, and let $T : Z \to X : (z_1, \ldots, z_n) \mapsto T_1z_1 + \ldots + T_nz_n$. Then T is a continuous linear operator from Z onto X so is an open mapping. Thus there is a constant M such that x in X implies that there is a z in Z with $\|z\| \le M\|x\|$ and $Tz = x$. The continuity of ST_j for all j gives the continuity of ST. For x in X, and z as above, we have $\|Sx\| = \|STz\| \le \|ST\|.M.\|x\|$, so S is continuous (see [151]).

We now consider the general case. Since $\mathfrak{S}(ST_j) \subseteq \mathfrak{S}(S)$ for each j we have just to prove that $\mathfrak{S}(S) \subseteq W$, where W is the closure of $\mathfrak{S}(ST_1) + \ldots + \mathfrak{S}(ST_n)$. If Q is the natural quotient operator from Y onto Y/W, then QST_j is continuous for each j so that QS is continuous by the previous paragraph. Hence $\mathfrak{S}(S) \subseteq W$ by Lemma 1.2 and the proof is complete.