1·Germs with constant rank

Literature: J. Dieudonné: Foundations of modern analysis, Academic Press (1969).

1.1. Suppose $A \subset \mathbf{R}^n$ is an arbitrary subset. A map $f : A \to \mathbf{R}^k$ is called differentiable if there is an open set $U \subset \mathbf{R}^n$ and a map $F : U \to \mathbf{R}^k$, such that $A \subset U$ and $F|A = f$, and such that the partial derivatives of F of every order exist and are continuous.

In what follows, our main interest will be in the local properties of maps. To make this more precise, we need the following definition: suppose $x \in A \subset \mathbf{R}^n$, V is a set, and $\mathfrak{F}$ is the set of pairs (U, f), where U is open in $\mathbf{R}^n$, $x \in U$, and $f : A \cap U \to V$ is an (arbitrary, continuous, differentiable, analytic, ...) map. Consider the following equivalence relation on the elements of $\mathfrak{F}$: for (U_1, f_1) and (U_2, f_2) in $\mathfrak{F}$, $(U_1, f_1) \sim (U_2, f_2)$ if and only if there is an open set $U \subset \mathbf{R}^n$, with $x \in U$ and $U \subset U_1 \cap U_2$, such that $f_1|U = f_2|U$. An equivalence class of this relation is called an (arbitrary, continuous, differentiable, analytic, ...) germ $\tilde{f} : (A, x) \to V$ at x (the tilde will frequently be omitted). Thus one speaks of germs of differentiable or analytic maps; further, since all subsets are defined by maps: $\mathbf{R}^n \to \{0, 1\}$, one may consider germs of subsets of $\mathbf{R}^n$. Germs behave much the same as maps, in particular, germs $\tilde{f}$, $\tilde{g}$ may be composed to give $\tilde{g} \circ \tilde{f}$:

$$(\mathbf{R}^n, x) \xrightarrow{\tilde{f}} (\mathbf{R}^m, y) \xrightarrow{\tilde{g}} \mathbf{R}^k$$

$$y = f(x)$$

$$\tilde{g} \circ \tilde{f} : (\mathbf{R}^n, x) \to \mathbf{R}^k .$$

If $f : U \to \mathbf{R}^m$, $g : V \to \mathbf{R}^k$ are representatives of $\tilde{f}$, $\tilde{g}$ then $f|f^{-1}(V) : f^{-1}(V) \to \mathbf{R}^m$ is a representative of $\tilde{f}$. The usual map-composite $g \circ f$ is defined on $f^{-1}(V) \subset U$ and this is a representative of

$\tilde{g} \circ \tilde{f}$. A differentiable germ $\tilde{f}: (\mathbf{R}^n, x) \to \mathbf{R}^k$ has a Jacobi-matrix $Df(x) : \mathbf{R}^n \to \mathbf{R}^k$ (a linear map). The germ $\tilde{f}$ has an inverse germ (with respect to '$\circ$') if and only if a representative f of $\tilde{f}$ has a local inverse in a sufficiently small neighbourhood of x. And this is the case if and only if $Df(x)$ is non-singular:

1.2. **Inverse-function theorem** (see Dieudonné). A germ $\tilde{f}: (\mathbf{R}^n, x) \to (\mathbf{R}^n, y)$ possesses an inverse germ $\tilde{f}^{-1} : (\mathbf{R}^n, y) \to (\mathbf{R}^n, x)$ if and only if the Jacobi-matrix $Df(x)$ is non-singular.

If $f : U \to \mathbf{R}^k$ is differentiable, $U \subset \mathbf{R}^n$, then the map $Df : U \to \mathbf{R}^{kn} = \{(k \times n)\text{-matrices}\}$, $x \mapsto Df(x)$ is differentiable. The rank of f at x is defined to be the rank of $Df(x)$ and denoted $Rk_x f$. If $Rk_x f \geq s$, then a certain $(s \times s)$-sub-matrix of $Df(x)$ has non-vanishing determinant. This determinant will be non-zero on a neighbourhood of x because Df and determinant are continuous. Hence the rank of f is never smaller than s on a neighbourhood of x, the rank of f cannot fall locally, and so the map $U \to \mathbf{Z}$, $x \mapsto Rk_x f$ is lower semicontinuous. Thus for any germ $\tilde{f}: (\mathbf{R}^n, x) \to \mathbf{R}^k$, there is a corresponding lower semi-continuous germ $Rkf : (\mathbf{R}^n, x) \to \mathbf{Z}$, $y \mapsto Rk_y f$.

One important consequence which we shall deduce from the inverse-function theorem is the following:

1.3. **The rank theorem** (see Dieudonné). Let $\tilde{f}: (\mathbf{R}^n, x) \to (\mathbf{R}^m, y)$ be a germ with constant rank (this means that the germ Rkf is the germ of a constant map) then there exist invertible germs $\tilde{\phi} : (\mathbf{R}^n, x) \to (\mathbf{R}^n, 0)$ and $\tilde{\psi} : (\mathbf{R}^m, y) \to (\mathbf{R}^m, 0)$, such that the germ

$$\tilde{\psi} \circ \tilde{f} \circ \tilde{\phi}^{-1} : (\mathbf{R}^n, 0) \to (\mathbf{R}^m, 0)$$

is represented by the map $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_k, 0, \ldots, 0)$ where $k = Rk_x f$.

Forgetting about germs, this result says that if a map $f : U_1 \to \mathbf{R}^m$, defined on the neighbourhood U_1 of x, has constant rank on a possibly smaller neighbourhood U_2 of x, then on a still smaller neighbourhood U_3 of x the map f has the given form with respect to suitable coordinates on $\mathbf{R}^n$ and $\mathbf{R}^m$.

Proof. Without loss of generality $x = y = 0$. Suppose f is a representative of $\tilde{f}$, with constant rank k. There will be a $(k \times k)$-submatrix of Df which is regular at the origin. By change of coordinates, that is, by applying local diffeomorphisms (invertible differentiable maps) the submatrix

$$(\partial f_i/\partial x_j), \quad 1 \le i, \quad j \le k$$

may be assumed regular at $0 \in \mathbf{R}^n$, and hence regular on a neighbourhood of the origin.

Define the germ $\tilde{\phi} : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ by

$$(x_1, \ldots, x_n) \mapsto (f_1(x), \ldots, f_k(x), x_{k+1}, \ldots, x_n)$$

where f has components $(f_1, \ldots, f_m)$.

$$D\phi = \begin{array}{c} \left[\begin{array}{c|c} \multicolumn{2}{c}{\partial f_i/\partial x_j} \\ \hline 0 & \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \end{array}\right] \begin{array}{l} \}\, k \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} n-k \end{array} \\ \quad\underbrace{\qquad}_{k} \qquad \underbrace{\qquad}_{n-k} \end{array}$$

$$\det(D\phi) = \det(\partial f_i/\partial x_j)_{1 \le i,\, j \le k}$$
$$\ne 0.$$

Hence $\tilde{\phi}$ is an invertible germ and the diagram

$$\begin{tikzcd} (\mathbf{R}^n, 0) \arrow[rr, "f"] \arrow[dr, "\phi"'] & & (\mathbf{R}^m, 0) \\ & (\mathbf{R}^n, 0) \arrow[ur, "g = f \circ \phi^{-1}"'] & \end{tikzcd} \qquad \begin{tikzcd} (x_1, \ldots, x_n) \arrow[r, "f"] \arrow[d] & (f_1, \ldots, f_m) \\ (f_1, \ldots, f_k, x_{k+1}, \ldots, x_n) \arrow[ur] \arrow[d, equal] & \\ (z_1, \ldots \qquad , z_n) & \end{tikzcd}$$

shows that the germ $\tilde{g} = \tilde{f} \circ \tilde{\phi}^{-1}$ is represented by

$$z = (z_1, \ldots, z_n) \overset{\mapsto}{_g} (z_1, \ldots, z_k, g_{k+1}(z), \ldots, g_m(z)).$$

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The Jacobian matrix of g has the form

$$Dg = \left[\begin{array}{ccc|c} 1 & & 0 & \\ & \ddots & & 0 \\ 0 & & 1 & \\ \hline & ? & & A(z) \end{array}\right],$$

$$A(z) = (\partial g_j/\partial z_i) \quad k+1 \le j \le m, \ k+1 \le i \le n$$

and because $Rk(g) = Rk(Dg) = k$ in a neighbourhood of the origin, the matrix $A(z)$ must vanish on this neighbourhood. Hence, without loss of generality:

(*) $\quad \partial g_j/\partial z_i = 0$ for $k+1 \le j \le m$, $k+1 \le i \le n$.

Now apply a local transformation in the range $\mathbf{R}^m$, namely the germ $\tilde{\psi} : (\mathbf{R}^m, 0) \to (\mathbf{R}^m, 0)$ given by

$$\begin{bmatrix} y_1 \\ \vdots \\ y_k \\ y_{k+1} \\ \vdots \\ y_m \end{bmatrix} \mapsto \begin{bmatrix} y_1 \\ \vdots \\ y_k \\ y_{k+1} - g_{k+1}(y_1, \ldots, y_k, 0, \ldots, 0) \\ \vdots \\ y_m - g_m(y_1, \ldots, y_k, 0, \ldots, 0) \end{bmatrix}$$

The Jacobian of $\tilde{\psi}$ has the form

$$D\tilde{\psi} = \begin{array}{c} \begin{array}{cc} \overbrace{\qquad\quad}^{k} & \overbrace{\qquad\quad}^{m-k} \end{array} \\ \left[\begin{array}{ccc|ccc} 1 & & 0 & & 0 & \\ & \ddots & & & & \\ 0 & & 1 & & & \\ \hline & & & 1 & & 0 \\ & ? & & & \ddots & \\ & & & 0 & & 1 \end{array}\right] \end{array}$$

since $g_{k+1}(y_1, \ldots, y_k, 0, \ldots, 0)$ etc., do not depend on $y_{k+1}, \ldots, y_m$. Hence $\tilde{\psi}$ is invertible, and $\tilde{\psi} \circ \tilde{g}$ is represented by the composite

$$\begin{bmatrix} z_1 \\ \vdots \\ z_k \\ z_{k+1} \\ \vdots \\ z_n \end{bmatrix} \mapsto \begin{bmatrix} z_1 \\ \vdots \\ z_k \\ g_{k+1}(z) \\ \vdots \\ g_m(z) \end{bmatrix} \mapsto \begin{bmatrix} z_1 \\ \vdots \\ z_k \\ g_{k+1}(z) - g_{k+1}(z_1, \ldots, z_k, 0, \ldots, 0) \\ \vdots \\ g_m(z) - g_m(z_1, \ldots, z_k, 0, \ldots, 0) \end{bmatrix}.$$

Because of (*) the last $m - k$ components $g_{k+j}(z_1, \ldots, z_n) - g_{k+j}(z_1, \ldots, z_k, 0, \ldots, 0)$ of this composite vanish on an n-cube $|z_j| < \varepsilon$. Hence $\tilde{\psi} \circ \tilde{g}$ is represented by

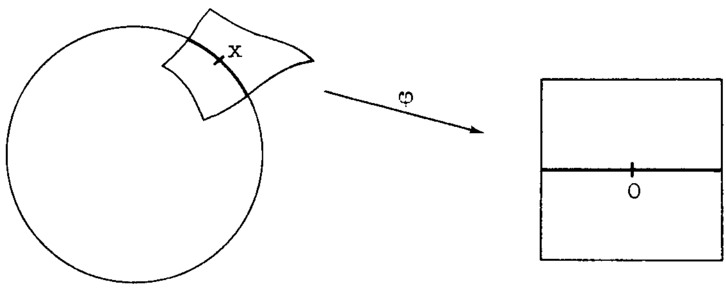
$$(z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_k, 0, \ldots, 0). \quad \checkmark$$

1.4. Application. Let $U \subset \mathbf{R}^n$ be an open subset, then $f: U \to \mathbf{R}^k$ is called a <u>submersion</u> if $\mathrm{Rg}_x f = k$ and an <u>immersion</u> if $\mathrm{Rg}_x f = n$ for all $x \in U$. By the rank theorem a submersion (immersion) has the form

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_k)$$
$$((x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0, \ldots, 0))$$

with respect to suitable coordinates. For, its rank cannot become any larger and is thus constant.

1.5. Definition. A subset $M \subset \mathbf{R}^n$ is called a <u>differentiable submanifold</u> of $\mathbf{R}^n$ of <u>dimension</u> $m \le n$, if to each $x \in M$ there corresponds an invertible germ $\tilde{\phi} : (\mathbf{R}^n, x) \to (\mathbf{R}^n, 0)$, such that $\tilde{\phi}(M, x) = (\mathbf{R}^m, x) \subset (\mathbf{R}^n, x)$ ($\mathbf{R}^m$ linearly imbedded in $\mathbf{R}^n$ for $m \le n$)



1.6. Example. $S^n = \{x \in \mathbf{R}^{n+1} | \langle x, x\rangle = 1\}$ is a differentiable submanifold of $\mathbf{R}^{n+1}$. The proof is left as an exercise.

1.7. Exercise. The set $LA(n, m; k) \subset LA(n, m) = \mathbf{R}^{nm}$ of $(m \times n)$-matrices with rank k in the space of all $(m \times n)$-matrices is a differentiable submanifold. Determine its dimension.

1.8. Definition. Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be differentiable. A point $y \in \mathbf{R}^m$ is called a regular value of f if at each $x \in \mathbf{R}^n$, such that $f(x) = y$, the rank of f is m, that is, $Rk_x f = m$. Any value of f which is not regular is said to be a critical value. If $y \notin f(\mathbf{R}^n)$, then this definition makes y a regular value of f.

1.9. Theorem. If y is a regular value of $f : \mathbf{R}^n \to \mathbf{R}^m$, then $f^{-1}\{y\} \subset \mathbf{R}^n$ is a differentiable submanifold of dimension $m - n$ (or empty).

Proof. Let $x \in f^{-1}\{y\}$, so that $f(x) = y$ and $Rk_x f = n$. This means that the rank of f is locally constant at x. By the rank theorem, there are local differentiable transformations $\phi : (\mathbf{R}^n, x) \to (\mathbf{R}^n, 0)$ and $\psi : (\mathbf{R}^m, y) \to (\mathbf{R}^m, 0)$ such that the germ $\tilde{\psi} \circ \tilde{f} \circ \tilde{\phi}^{-1} = \tilde{f}_1$ has the form

$$f_1(x_1, \ldots, x_m, \ldots, x_n) = (x_1, \ldots, x_m).$$

The germ $\tilde{f}_1^{-1}\{0\} = \tilde{\phi}\tilde{f}^{-1}\tilde{\psi}^{-1}\{0\} = \tilde{\phi}\tilde{f}^{-1}\{y\}$ is the germ of the set $\{(0, \ldots, 0, x_{m+1}, \ldots, x_n)\}$ at the origin. ✓

1.10. Exercises.

1. Suppose that A is a symmetric $(n \times n)$-matrix and $0 \neq b \in \mathbf{R}$, prove that the set $M = \{x \in \mathbf{R}^n | x^t A x = b\}$ is either an $(n-1)$-dimensional submanifold of $\mathbf{R}^n$ or empty.

2. Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a differentiable map such that $f \circ f = f$. Prove that $f(\mathbf{R}^n) \subset \mathbf{R}^n$ is a differentiable submanifold.

3. In general, $f^{-1}\{y\}$ is not necessarily a submanifold - give a counter example.

If $M^m \subset \mathbf{R}^{m+k}$ is a differentiable submanifold and if we choose an invertible representative for every germ $\tilde{\phi}$ in the definition of differentiable submanifold, then:

Each $x \in M$ has a neighbourhood U_λ on which a differentiable map $\phi_\lambda : U_\lambda \overset{\cong}{\rightarrow} U'_\lambda \subset \mathbf{R}^m$ is defined, where U'_λ is an open subset.

If one wants to define a differentiable manifold without using an imbedding in $\mathbf{R}^{m+k}$, then the following definition presents itself:

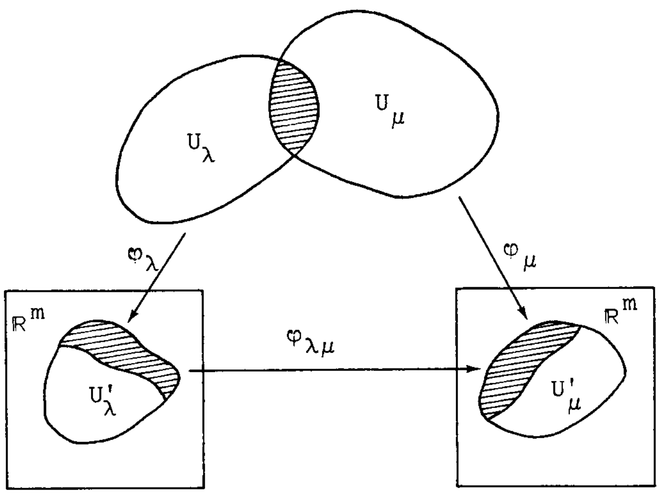
1.11. A <u>differentiable manifold</u> is a topological space M, together with an open cover $\{U_\lambda \mid \lambda \in \Lambda\}$ and homeomorphisms $U_\lambda \rightarrow U'_\lambda \subset \mathbf{R}^m$ (U'_λ open), with the following properties:

(I)

$$\begin{array}{ccc} U_\lambda & \cap & U_\mu \\ \downarrow{\scriptstyle \phi_\lambda} & & \downarrow{\scriptstyle \phi_\mu} \\ \mathbf{R}^m \supset U'_{\lambda\mu} & \xrightarrow[\phi_{\lambda\mu}]{} & U'_{\mu\lambda} \subset \mathbf{R}^m \end{array},$$

if $U_\lambda \cap U_\mu \neq \emptyset$, the map $\phi_{\lambda\mu}$ exists to make the diagram commute and is differentiable. Since $\phi_{\lambda\mu}\phi_{\mu\lambda} = \mathrm{Id}$, the map $\phi_{\lambda\mu}$ is a diffeomorphism (invertible). Obviously $U'_{\lambda\mu} = \phi_\lambda(U_\lambda \cap U_\mu)$ etc.

(II) M is Hausdorff and has a countable basis.



Many concepts are carried over from euclidean space to submanifolds of $\mathbf{R}^n$ (and to manifolds). For example, let $x \in M^m \subset \mathbf{R}^{m+k}$ then a function $f : M \rightarrow \mathbf{R}$ is <u>differentiable</u> at x if there is an invertible germ

$\tilde{\psi} : (\mathbf{R}^{m+k}, x) \to (\mathbf{R}^{m+k}, 0)$ such that

$$\tilde{\psi} : (M^m, x) \xrightarrow{\cong} (\mathbf{R}^m, 0)$$

and such that the germ $\tilde{f} \circ \tilde{\psi}^{-1}$ is differentiable at $0 \in \mathbf{R}^m \subset \mathbf{R}^{m+k}$.

In other words: M is covered by open sets U_λ, which may be identified with open subsets of $\mathbf{R}^m$ using coordinate transformations. A map defined on M is called differentiable (has rank r, etc.) if restricted to these open subsets - i. e. subsets of $\mathbf{R}^m$ up to transformation - it is differentiable (has rank r, etc.).

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2 · Regular values

Literature: J. Milnor: Topology from the differentiable viewpoint, Virginia (1969).
R. Narasimhan: Analysis on real and complex manifolds, Masson and Cie, Paris, and North-Holland, Amsterdam (1968).
S. Sternberg: Lectures on differential geometry, Prentice-Hall (1964).

The aim of this chapter is to prove the following theorem:

2.1. **Sard's theorem.** The Lebesgue-measure of the set of critical values of a differentiable map is zero.

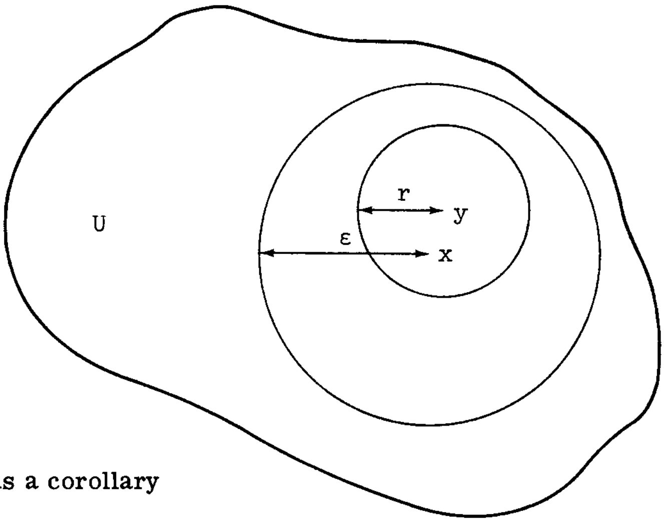
A set with measure zero is defined below, but first of all consider an application: let $f : \mathbf{R}^n \to \mathbf{R}^m$ be differentiable. Then for almost all points $b \in \mathbf{R}^m$ (i.e. everywhere except for a set with Lebesgue measure zero) the following is true: $f^{-1}\{b\} \subset \mathbf{R}^n$ is a differentiable submanifold of dimension $n - m$. In other words:

For given $f = (f_1, \ldots, f_m)$ and for almost any choice of $b_i \in \mathbf{R}$, $1 \le i \le m$, the system of non-linear equations $f_i(x) = b_i$, $x \in \mathbf{R}^n$, $1 \le i \le m$, has an (n-m)-dimensional manifold for its solution set.

2.2. **Preliminaries.** Let $\{K_n\}_{n \in \mathbf{N}}$ be the set of balls in $\mathbf{R}^m$ with rational radius and rational coordinates at the centre (there are countably many!).

If $U \subset \mathbf{R}^m$ is open, then $U = \bigcup_{i \in T} K_i$ for a certain subset $T \subset \mathbf{N}$.

Proof. Let $x \in U$ and ε be small enough so that the ε-neighbourhood of x is contained in U. Choose K_i with centre at y for $|x - y| < \varepsilon/3$ and rational radius r, where $|x - y| < r < 2\varepsilon/3$. ✓



As a corollary

2.3. **Remark.** Let $X \subset \mathbf{R}^n$ be an arbitrary subset and $\{U_\lambda\}_{\lambda\in\Lambda}$ a family of open subsets of $\mathbf{R}^n$, such that $X \subset \bigcup_{\lambda\in\Lambda} U_\lambda$, then there is a countable subset $\Gamma \subset \Lambda$ such that $X \subset \bigcup_{\lambda\in\Gamma} U_\lambda$.

Proof. X is in the union of those K_n, which are contained in at least one U_λ, and there are only countably many such K_n. For each of these K_n choose a corresponding $U_{\lambda(n)}$ where $K_n \subset U_{\lambda(n)}$. The set X is contained in the union of the $U_{\lambda(n)}$. ✓

2.4. **Definition.** A subset $C \subset \mathbf{R}^n$ has measure zero, if for each $\varepsilon > 0$, there exists a sequence of cubes $W_i \subset \mathbf{R}^n$ such that

$$C \subset \bigcup_{i=1}^{\infty} W_i, \quad \text{and} \quad \sum_{i=1}^{\infty} |W_i| < \varepsilon.$$

Here $|W_i|$ is the volume of W_i, that is, $|W_i| = a^n$ where a is the side-length of W_i.

2.5. Clearly if $C = \bigcup_{\nu=1}^{\infty} C_\nu$ and each C_ν has measure zero, then C also has measure zero. For,

$$C_\nu \subset \bigcup_{i=1}^{\infty} W_i^\nu \quad \text{where} \quad \sum_{i=1}^{\infty} |W_i^\nu| < \frac{\varepsilon}{2^\nu}$$

and this implies