I · Preliminaries

1. Historical notes

Abstract topological groups were first defined by Schreier in 1926, though the idea was implicit in much earlier work on continuous groups of transformations. The subject has its origins in Klein's programme (1872) to study geometries through the transformation groups associated with them, and in Lie's theory of continuous groups arising from the solution of differential equations. The 'classical groups' of geometry (general linear groups, unitary groups, symplectic groups, etc.) are in fact Lie groups, that is, they are analytic manifolds and their group operations are analytic functions. On the other hand, Killing and Cartan showed (1890) that all simple Lie groups are classical groups, apart from a finite number of exceptional groups.

In 1900 Hilbert posed the problem (No. 5 of his famous list) whether every continuous group of transformations of a finite-dimensional real or complex space is a Lie group. The twentieth-century habit of axiomatising everything led to a more abstract formulation of this problem. A topological group is a topological space which is a group with continuous group operations, and the question is: What topological conditions on a topological group will ensure that it has an analytic structure which makes it a Lie group? Since integration was a major tool in the study of Lie groups, especially their representations, it became important to establish the existence of appropriate integrals on general classes of topological groups. This was achieved by Haar in 1933 for locally compact groups with countable open bases. Von Neumann (1934) gave another proof for arbitrary compact groups, making the representation theory of compact Lie groups immediately available for all compact groups and so solving Hilbert's problem in this special case. Haar's method of integration was extended to all locally compact groups by Weil (1940). However, there are serious obstacles to extending the representation theory to locally compact groups, and it was not until 1952 that Hilbert's problem was settled by Gleason, Montgomery and Zippin. Their answer can be formulated as follows: a topological group is a Lie group if and only if it is locally Euclidean; alternatively, it is a Lie group if and only if it is locally compact and does not have arbitrarily small subgroups, that is, the identity element has a compact neighbourhood containing no non-trivial subgroups.

Although the theory of topological groups was developed mainly in order to study groups of Lie type and its impetus came from problems in analysis, it soon proved to be useful also in purely algebraic contexts. Certain algebraic constructions lead to groups having natural topological structures which are somewhat pathological from an analyst's point of view. Examples are power-series rings, Galois groups of infinite field extensions, and p-adic groups. The pathology lies in the existence of arbitrarily small subgroups, but in most important cases the groups are actually locally compact and integration is therefore possible on them. The algebraist must be familiar with these facts and this course is designed to make them available to him. It will, I hope, also serve as an introduction to topological groups and the Haar integral for students of other branches of mathematics. However, the general flavour of the development is more algebraic than is usual in such an introduction, and the use of analytical arguments has been kept to a minimum. The only pre-requisites for the first three chapters are a few facts from elementary group theory. Chapter IV is less rigorous and demands more of the reader.

2. Categories

Definition. A <u>category</u> $\mathcal{C}$ is a structure comprising the following data:

(i) a class whose members A, B, C, ... are called the <u>objects</u> of $\mathcal{C}$;

(ii) for each pair of objects A, B, a set $\mathcal{C}(A, B)$, called the set of <u>morphisms from</u> A <u>to</u> B (in $\mathcal{C}$); we write $f : A \to B$ to mean that $f \in \mathcal{C}(A, B)$;

(iii) for each triple of objects A, B, C, a law of composition

$$\mathcal{C}(A, B) \times \mathcal{C}(B, C) \to \mathcal{C}(A, C);$$

that is, for $f : A \to B$ and $g : B \to C$, there is defined a 'composite' morphism $fg : A \to C$. (Note: we have adopted a right-handed notation for morphisms which is contrary to the current practice of many authors.)

These data are subject to the following axioms:

I. Associative Law. If $f : A \to B$, $g : B \to C$ and $h : C \to D$, then $(fg)h = f(gh)$.

II. Identities. For each object A, there is a morphism $e_A \in \mathcal{C}(A, A)$ such that for all $f : A \to B$, $e_A f = f$, and for all $g : C \to A$, $ge_A = g$.

Examples. (i) The category $\mathcal{S}$ of sets. Its objects $A, B, C, \ldots$ are sets, and the morphisms $f : A \to B$ are functions (maps) from A to B. Composition is ordinary composition of functions. The identities are the maps

$$e_A = \iota_A : a \mapsto a, \text{ for } a \text{ in } A.$$

(Note: in any category in which the morphisms are certain maps between sets, and composition is the ordinary one of maps, associativity follows immediately.)

(ii) The category $\mathcal{G}$ of groups. Its objects are groups, and the morphisms $A \to B$ are group homomorphisms. Composition and identities are as in $\mathcal{S}$. (The point is that if f, g are group homomorphisms, so is fg; and ι_A is always a group homomorphism.)

(iii) The category $\mathcal{T}$ of topological spaces and continuous mappings, defined as follows: Given a set X, a topology $\mathcal{U}$ on X is a collection of subsets of X, called 'open subsets of X', satisfying:

(a) the intersection of two open sets is open (whence so is any finite intersection),

(b) an arbitrary union of open sets is open, and

(c) the empty set $\emptyset$, and X itself, are open subsets of X.

We then call $(X, \mathcal{U})$ a topological space. Notation: if $\mathcal{U}$ is fixed, we shall say simply 'X is a topological space'.

The objects of $\mathcal{T}$ are the topological spaces. Their morphisms are mappings $f : A \to B$ (A, B topological spaces) which are continuous, i.e. such that for all U open in B, the pre-image $Uf^{-1} (= \{x \in A;\ xf \in U\})$

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is open in A. Composition and identities are as in $\mathcal{S}$. (Check: f, g continuous $\Rightarrow$ fg continuous.)

The language of categories enables one to define certain familiar concepts very generally. This is helpful in comparing and contrasting analogous situations in different mathematical contexts.

Definition. Let $\mathcal{C}$ be a category. Then $f : A \to B$ in $\mathcal{C}$ is an isomorphism (in $\mathcal{C}$) if it is invertible in $\mathcal{C}$, i.e. if it has an inverse $g : B \to A$ in $\mathcal{C}$ such that $gf = e_B$, $fg = e_A$. If such a morphism g exists, it is unique, and we write $g = f^{-1}$. We also say that A and B are $\mathcal{C}$-isomorphic and write $A \underset{\mathcal{C}}{\cong} B$.

Examples. The isomorphisms in $\mathcal{S}$ are the bijections (1 - 1 correspondences).

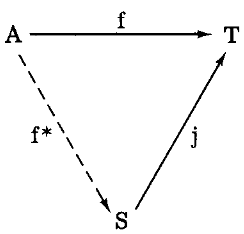
The isomorphisms in $\mathcal{G}$ are the group isomorphisms (bijective group homomorphisms). If a group homomorphism f is bijective, its set-inverse f^{-1} is automatically a group homomorphism.

The isomorphisms in $\mathcal{T}$ are the homeomorphisms, i.e. bijective maps f such that f is continuous and f^{-1} is continuous. There are continuous bijections in $\mathcal{T}$ which are not isomorphisms, i.e. their set-inverses are not continuous.

Salient facts about $\mathcal{S}$. Subsets: If $S \subseteq T$, the 'inclusion' map $j : S \to T$ in $\mathcal{S}$, $(sj = s)$ is an injection and has the following universal property:

if $f : A \to T$ in $\mathcal{S}$, and $Af \subseteq S$, then $\exists!\, f^* : A \to S$ such that $f = f^*j$.

Pictorially:



$\exists!\ f^*$ such that the diagram commutes.

($\exists!$ means 'there exists a unique...')

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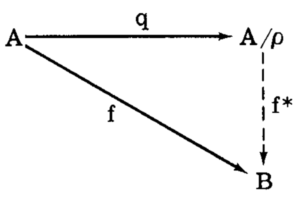
Quotient sets: If A, B are sets, a correspondence from A to B is a subset γ of $A \times B$. If also δ is a correspondence from B to C, $\delta \subseteq B \times C$, we obtain a correspondence $\gamma\delta \subseteq A \times C$ by letting $(a, c) \in \gamma\delta$ if and only if $\exists\, b \in B$ such that $(a, b) \in \gamma$, $(b, c) \in \delta$. We also define $\gamma^{-1} \subseteq B \times A$ by the rule: $(b, a) \in \gamma^{-1}$ if and only if $(a, b) \in \gamma$. The identity correspondence $\iota_A : A \to A$ is $\{(a, a);\ a \in A\}$. In this notation a function $f : A \to B$ is a correspondence such that $ff^{-1} \supseteq \iota_A$, and $f^{-1}f \subseteq \iota_B$.

An equivalence relation on a set A is a correspondence $\gamma \subseteq A \times A$ such that $\gamma \supseteq \iota_A$ (γ is reflexive), $\gamma = \gamma^{-1}$ (γ is symmetric) and $\gamma\gamma \subseteq \gamma$ (γ is transitive). Given an equivalence relation ρ on A, A is partitioned into equivalence classes $(a) = \{b \in A;\ (a, b) \in \rho\}$. The quotient A/ρ is the set of equivalence classes, and there is a canonical map $q : A \to A/\rho$, $a \mapsto (a)$. This q is a surjection ('an onto map').

For $f : A \to B$, ff^{-1} is an equivalence relation on A whose classes are the 'fibres' of f, namely the sets of form bf^{-1}, for $b \in Af$.

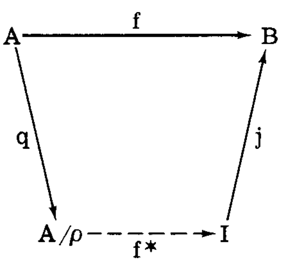
Definition. ff^{-1} is the kernel of f, denoted by $\mathrm{Ker}(f)$.

The quotient map $q : A \to A/\rho$, where ρ is an equivalence relation, has the following universal property: if $f : A \to B$ and $\rho \subseteq \mathrm{Ker}(f) = ff^{-1}$, $\exists!\, f^* : A/\rho \to B$ such that $qf^* = f$.



$\exists!\, f^*$ such that the diagram commutes.

Proposition 1 ($\mathcal{S}$). (First isomorphism theorem.) Let $f : A \to B$ be any map. Let ρ be its kernel with quotient map $q : A \to A/\rho$. Let



$I = Af \subseteq B$, with inclusion $j : I \to B$. Then (by the universal properties for quotients and subsets), $\exists!\ f^* : A/\rho \to I$ such that $qf^*j = f$. Further, f^* is an isomorphism in $\mathcal{S}$.

Products in $\mathcal{S}$: If I, S are any sets and $x : I \to S$ any map, we often speak of x as a family (indexed set) of elements in S. We use the notation $\{x_i\}_{i \in I}$ for the family, where $x_i = ix$, and we call I the indexing set of the family. For example a sequence of real numbers $(x_n)_{n \in \mathbf{N}}$ is a function $x : \mathbf{N} \to \mathbf{R}$. If $\{A_i\}_{i \in I}$ is a family of sets, we define:

$$\prod_{i \in I} A_i = \{\{a_i\}_{i \in I};\ \forall\, i \in I,\ a_i \in A_i\}.$$

If I is finite, $I = \{1, \ldots, n\}$, then $\prod_{i \in I} A_i$ is $\mathcal{S}$-isomorphic with $A_1 \times A_2 \times \ldots \times A_n$.

Suppose that $a = \{a_i\}_{i \in I}$ is a member of $A = \prod_{i \in I} A_i$. We call a_i the i^{th} coordinate of a, and $\pi_i : A \to A_i$, $a \mapsto a_i$ is called the i^{th} coordinate projection. These projections have a universal property: if for each $i \in I$ we are given a map $f_i : B \to A_i$, then $\exists!\ f : B \to A$ such that $f\pi_i = f_i$ for each i. (This map is given by $bf = \{bf_i\}_{i \in I}$.)

We are thus led to the categorical concept of product. Let $\mathcal{C}$ be a category. Suppose that I is a set, and for each $i \in I$ we are given an object A_i of $\mathcal{C}$. Suppose that A is also an object of $\mathcal{C}$, with morphisms $\pi_i : A \to A_i$ for $i \in I$. We say that A is a product in $\mathcal{C}$ of the A_i, with projections π_i, if for every family of morphisms $\{f_i\}_{i \in I}$, with $f_i : B \to A_i$ in $\mathcal{C}$, $\exists!\ f : B \to A$ in $\mathcal{C}$, such that $f\pi_i = f_i$ for every $i \in I$.

Such a product need not exist. However, any two such products must be isomorphic in $\mathcal{C}$ by the 'standard argument' for universal properties, which we write out in full here (it will be left as an easy exercise at all future occurrences).

If A is a product of the objects A_i, with projections π_i, let us (temporarily) write $f_i, \pi_i \rightsquigarrow f$ to mean that the morphisms $f_i : B \to A_i$ determine (uniquely) the morphism $f : B \to A$ satisfying $f\pi_i = f_i$ for all $i \in I$. If A' is another product of the A_i, with projections $\pi'_i : A' \to A_i$, let

$$\pi_i,\ \pi'_i \rightsquigarrow \alpha \quad\text{and}\quad \pi'_i,\ \pi_i \rightsquigarrow \beta .$$

Then $\alpha : A \to A'$ satisfies $\alpha\pi'_i = \pi_i$, and $\beta : A' \to A$ satisfies $\beta\pi_i = \pi'_i$. It follows that $\alpha\beta\pi_i = \pi_i$, for all $i \in I$, and hence

$$\pi_i,\ \pi_i \rightsquigarrow \alpha\beta .$$

But clearly, $\pi_i,\ \pi_i \rightsquigarrow \iota_A$, and it follows from the uniqueness clause that $\alpha\beta = \iota_A$. Similarly $\beta\alpha = \iota_A$. Thus $A \underset{\mathcal{C}}{\cong} A'$ and since $\alpha\pi'_i = \pi_i$ and $\beta\pi_i = \pi'_i$, the isomorphism respects the projections.

We may now say that $\mathcal{S}$ has products (i. e. any family of sets has a product in $\mathcal{S}$).

We shall always assume the Axiom of Choice: 'If the sets A_i $(i \in I)$ are non-empty, their product is also non-empty.' This is equivalent (given the other axioms of standard set theory) to Zorn's Lemma.

Definition. A partially ordered set is a set S together with a correspondence $\gamma \subseteq S \times S$ such that: $\gamma \supseteq \iota_S$ (γ is reflexive), $\gamma\gamma \subseteq \gamma$ (γ is transitive), $\gamma \cap \gamma^{-1} \subseteq \iota_S$ (γ is anti-symmetric). We usually write $a \leq b$ to mean $(a, b) \in \gamma$.

Let S denote a fixed partially ordered set.

Definition. $T \subseteq S$ is a chain if $\forall t, t' \in T$, either $t \leq t'$ or $t' \leq t$ (i. e. T is totally ordered by the induced relation $\gamma \cap T^2$.)

Definition. $x \in S$ is maximal in S if $\forall y \in S$, $y \geq x \Rightarrow y = x$.

Definition. $T \subseteq S$ is bounded above in S if $\exists s \in S$ such that $\forall t \in T$, $t \leq s$.

Definition. If every chain in S is bounded above in S, we say that S is inductively ordered.

Zorn's Lemma asserts: 'Every inductively ordered set has a maximal element.'

N. B., it is not enough to know that all countable ascending chains

$(s_1 < s_2 \ldots < s_n \ldots)$ are bounded above. E.g., take S = all countable subsets of $\mathbf{R}$, ordered by $\subseteq$. Any countable union of sets in S is also in S, hence countable ascending chains are bounded above in S (by their unions). However, no set in S can be maximal because, $\mathbf{R}$ being uncountable, we can always adjoin one more element.

3. Groups

We state some familiar facts about groups in categorical language.

Subgroups. Let G be a group. Let H be a subset of G with a group structure defined on it; then H is a subgroup of G if and only if the inclusion map $j : H \to G$ is a morphism of groups. There is at most one group structure on a given subset H for which this is the case. Subgroup inclusion $j : H \to G$ has a universal property, as for sets: if $f : L \to G$ is a morphism in $\mathcal{G}$ such that $Lf \subseteq H$, then $\exists !\ f^* : L \to H$ in $\mathcal{G}$ such that $f^*j = f$.

Quotient groups. For a morphism $f : G \to H$ of groups it is usual to define the kernel of f as the normal subgroup $\{x \in G;\ xf = e\}$ of G, where $e = e_H$ denotes the identity element of H. This is in conflict with our earlier definition in $\mathcal{S}$, namely, $\operatorname{Ker} f = ff^{-1}$, but the two are closely related, each determining the other. (If we write $K = \{x \in G;\ xf = e\}$, then K is the equivalence class of ff^{-1} containing e_G. On the other hand, ff^{-1} is just the equivalence relation on G whose classes are all the cosets of K.) There is no danger in using the same name, Ker f, for both concepts - the context will always make it clear which is intended. For any normal subgroup N or F, the cosets of N are the equivalence classes of the equivalence relation ρ defined by $a\rho b \Longleftrightarrow ab^{-1} \in N$. It is usual to write G/N for the quotient set G/ρ in this case. If $q : G \to G/N$ is the corresponding quotient map, then there is a unique group structure on G/N such that q is a morphism of groups. We shall always give G/N this structure. The quotient map $q : G \to G/N$ in $\mathcal{G}$ has a universal property, as for sets: if $f : G \to H$ is a morphism in $\mathcal{G}$ such that $N \subseteq \operatorname{Ker} f$ (i.e., $Nf = \{e\}$), then $\exists !\ f^* : G/N \to H$ in $\mathcal{G}$ such that $qf^* = f$.

Proposition 1 ($\mathcal{G}$). (First isomorphism theorem.) For $f : G \to H$ in $\mathcal{G}$, let $K = \operatorname{Ker} f$, and let $I = Gf$ (a subgroup of H). Let $q : G \to G/N$

be the quotient map and $j : I \to H$ the inclusion map. Then $\exists!\ f^* : G/N \to I$ (in $\mathcal{S}$) such that $qf^*j = f$. Furthermore, f^* is a $\mathcal{G}$-isomorphism.

Products in $\mathcal{G}$. Suppose we are given for each $i \in I$ a group A_i with identity e_i. Form $A = \Pi A_i$ in $\mathcal{S}$. Define $e_A = \{e_i\}$, and for $a = \{a_i\}$, $b = \{b_i\} \in A$, define $a^{-1} = \{a_i^{-1}\}$, $ab = \{a_i b_i\}$. (We note that $a_i, b_i \in A_i \Rightarrow a_i^{-1}, a_i b_i \in A_i$.) Then with these operations A is a group with identity e_A, and the projections $\pi_i : A \to A_i$ are group-morphisms.

If now $f_i : B \to A_i$ are group-morphisms, $\exists!\ f : B \to A$ (of sets) such that $f\pi_i = f_i$; it is given by $bf = \{bf\pi_i\} = \{bf_i\}$. One verifies that f is in fact a group-morphism, and hence that A is the product in $\mathcal{G}$ of the A_i.

4. Topological spaces

Examples of topological spaces:

(i) Any metric space X under the distance topology, i.e., $Y \subseteq X$ is open if and only if $\forall y \in Y$, $\exists \delta > 0$ such that the δ-ball with centre y is contained in Y. (E.g. $\mathbf{R}^n$ with the usual metric.)

(ii) Any set X with the *discrete* topology (all subsets are open; this is the strongest topology on X, where if S_1, S_2 are two topological spaces with underlying set X, we say S_1 is *stronger* than S_2, or S_2 is *weaker* than S_1, if the identity: $S_1 \to S_2$ is continuous, and strictly so if in addition the identity: $S_2 \to S_1$ is *not* continuous). *Warning*: some authors use the terms *weaker* and *stronger* in the opposite sense.

(iii) Any set X with the *trivial* topology ($\emptyset$, X are the only open sets; this is the weakest topology on X).

If X is a space with the discrete topology, or Y is a space with the trivial topology, then any mapping $f : X \to Y$ is continuous.

Subspaces. Let X be a topological space and $Y \subseteq X$ a subset. Consider all sets of the form $A \cap Y$, with A open in X. These may be taken as the open subsets of a topology on Y, the *subspace (or induced) topology* on Y. It is the weakest topology such that the inclusion map $j : Y \to X$ is continuous. 'Subspace' will always mean 'subset with the induced topology'.

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Example. The topological n-sphere S^n is defined to be the subset $\{\mathbf{x}; \sum_{i=1}^{n+1} x_i^2 = 1\}$ of $\mathbf{R}^{n+1}$ with the subspace topology, $\mathbf{R}^{n+1}$ having the usual metric topology.

Let Y be a subspace of X. Then $j : Y \to X$ has the usual universal property: if $f : Z \to X$ in $\mathcal{T}$ and $Zf \subseteq Y$, then $\exists!\ f^* : Z \to Y$ in $\mathcal{T}$ such that $f^*j = f$. (For we may construct f^* in $\mathcal{S}$, and continuity follows from the observation that, if A is open in X, then $(A \cap Y)f^{*-1} = (Aj^{-1})f^{*-1} = Af^{-1}$ is open in Z.)

Exercise. The subspace topology on Y is the only one for which j has this property (standard argument; Y is then determined up to $\mathcal{T}$-isomorphism).

Note the contrast with $\mathcal{G}$; a subset of a group G need not possess any group-structure making j a group-morphism, but if one exists, it is unique; whereas a subset of a space X always has a topology (which is not unique) making j continuous. Similar remarks apply to quotients in $\mathcal{T}$ which we now define.

Quotient spaces. Let ρ be an equivalence relation on a topological space X, and let $q : X \to X/\rho$ be the quotient map. The quotient (or identification) topology on X/ρ is the strongest one for which q is continuous, i.e., $A \subseteq X/\rho$ is open if and only if Aq^{-1} is open in X. By 'quotient (or identification) space' we shall always mean 'quotient set with the quotient topology', and X/ρ will always denote this space. In this situation, the quotient map q has the universal property that if $f : X \to Z$ is continuous and $\rho \subseteq \operatorname{Ker} f$, then $\exists!\ f^* : X/\rho \to Z$ in $\mathcal{T}$ such that $qf^* = f$. The quotient topology is the only topology on X/ρ such that q has this property. (The proof of these facts is left as an exercise.)

Example. On $\mathbf{R}$ with the usual topology, define ρ by the rule: $(a, b) \in \rho \iff a - b \in \mathbf{Z}$. Then $\mathbf{R}/\rho = \mathbf{R}/\mathbf{Z}$ is a group and, with the quotient topology, is homeomorphic to the subspace S^1 of $\mathbf{R}^2$, via the canonical bijection $\mathbf{Z} + t \mapsto (\cos 2\pi t, \sin 2\pi t)$. (Exercise: Show that this is a homeomorphism.)

N. B. The first isomorphism theorem is false in $\mathcal{T}$. For example, the identity map $S_1 \to S_2$, where S_1 is a stronger topology than S_2 over