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H. P. F. Swinnerton-Dyer
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Introduction

For fixed $n > 0$ let $\mathbf{C}^n$ denote n-dimensional complex space and $\mathbf{R}^{2n}$ the underlying 2n-dimensional real space. Let Λ be a lattice in $\mathbf{R}^{2n}$ - that is, a free abelian group on 2n generators which spans $\mathbf{R}^{2n}$; thus with the induced topology Λ is discrete and $T = \mathbf{R}^{2n}/\Lambda$ is compact. T has a natural structure as a complex manifold, which we recognize by writing it as $T = \mathbf{C}^n/\Lambda$, and with this structure it is called a complex torus. The most interesting case is when there are sufficiently many meromorphic functions on T, in a sense that will be made precise in Chapter II; in this case T is called an abelian manifold, and is usually denoted by A. Thus from one point of view the study of abelian manifolds is essentially the study of meromorphic functions of n complex variables having 2n independent periods. Thus it forms a natural generalization of the theory of elliptic functions - that is, of doubly periodic functions of one complex variable, Historically, the other parent of the subject was the study of compact Riemann surfaces; indeed the term 'abelian manifolds' comes from the connection with Abel's theorem.

A compact Riemann surface is just another way of describing a non-singular algebraic curve; and this already gives a connection between algebraic geometry and abelian manifolds. This connection was much strengthened by Lefschetz and the great Italian geometers, who showed that abelian manifolds are an important tool in the problem of classifying non-singular varieties. Conversely any abelian manifold can be embedded in projective space as a variety in the sense of algebraic geometry; when we wish to emphasize this point of view, we speak of an abelian variety instead of an abelian manifold. The study of abelian varieties by purely geometric methods, valid over fields of arbitrary characteristic, was initiated by Weil; see [14], [15] and Lang [5]. (Numbers in square brackets refer to the list of references at the end of this volume.) An up-to-date account may be found in Mumford [8]. More recently Shimura and others

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have shown that the theory of abelian manifolds has important application to algebraic number theory; for Shimura's work see [9], [10] and a series of papers published during the 1960s, largely in Annals of Mathematics.

The main object of this book is to give an account of the standard theory of abelian manifolds which presupposes not much more than a basic complex variable course. It contains all the material on abelian manifolds which is needed for the applications to algebraic geometry and to number theory; indeed it is based on a course of lectures delivered in Cambridge which was designed to lead on to an exposition of some of Shimura's work. But it does not contain an exposition of either of these applications. I have included some geometrical results, and they do presuppose a knowledge of algebraic geometry; the reader who lacks this may omit them without inconvenience. Also §10, on the structure of the ring of endomorphisms of an abelian variety, requires some knowledge of the theory of algebras and of algebraic number theory; however the necessary results from the theory of algebras are stated without proof at the beginning of §10.

The book starts with two sections closely connected with the main theme though not essential to it. In §1 there is a survey of the theory of compact Riemann surfaces - without proofs of the key theorems because to include proofs would take so much space that it would unbalance the book. Proofs may be found for example in Gunning [4]. In §2 there is a brief account, with proofs, of the theory of elliptic functions. This is the special case $n = 1$ of the theory, but it is untypical for two reasons. Because one can give a simple description of divisors when $n = 1$, there are many explicit formulae in that case which cannot be generalized; also the Riemann form, which plays a major part in the general theory, appears in so trivial a form when $n = 1$ that its presence goes unnoticed. The rest of Chapter I contains some general results on functions of several complex variables which are needed later. Chapter II is concerned with necessary and sufficient conditions on Λ for $T = \mathbf{C}^n/\Lambda$ to admit non-constant meromorphic functions, or to be an abelian manifold, and with the construction of those functions. It also contains the immediate consequences of these results, in particular theorems about the projective

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embedding of abelian manifolds. Chapter III contains the standard theory of abelian manifolds themselves, in particular the study of the ring of endomorphisms of an abelian manifold and the matrix representations of these endomorphisms, and the duality theory which essentially describes the group of divisors on an abelian manifold. Many of the results in this Chapter can be stated in purely algebro-geometric terms, even though the proofs which have been given for them are analytic. I have therefore given in an Appendix a brief account without proofs of the geometric theory of abelian varieties over a field of arbitrary characteristic, to show how far the analytic theory remains valid.

The most elegant account of the analytic theory is that given in Chapter VI of Weil [17]; but that account was written to show how Hodge theory can be applied to a particularly simple kind of complex manifold and it therefore assumes a great deal of background knowledge. The same is true of the account in Mumford [8], Chapter I. Two books which assume no more knowledge than this one are Conforto [2] and Lang [6]. Conforto's treatment is very old-fashioned; in particular a substantial part of his book is taken up by Poincaré's original proof of the key existence theorem, that to every positive divisor on $\mathbf{T}$ corresponds a theta-function on $\mathbf{C}^n$, whereas the proof given here, based on Weil [16], only takes a few pages. Lang's book is very similar in spirit to this one, though perhaps more modern; but I believe there is enough difference in the material covered to justify publishing this also. Much of the theory can also be found in Siegel [11].

Nothing in this book is original, except perhaps the errors. I have therefore only ascribed a result to someone if it is generally known by his name.

I would like to express my gratitude to André Weil, Serge Lang and S. J. Patterson for the help which they have given me at various levels; without them this book would never have been written.

NOTATION

As is usual, $\mathbf{C}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{Z}$ denote respectively the complex numbers, the rationals, the reals and the rational integers; but Q is sometimes also used to denote a quadratic form. From the beginning of

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Chapter II onwards, $V = \mathbf{C}^n$ is a complex vector space of dimension n and Λ is a lattice in V; the complex torus V/Λ is denoted by T in general, and by A when it is known to be an abelian manifold. In the Appendix, A is an abelian variety of dimension n defined over an arbitrary field k, not necessarily of characteristic zero. Both

$$z = (z_1, \ldots, z_n) \quad \text{and} \quad w = (w_1, \ldots, w_n)$$

usually denote points of V and their coordinates with respect to a fixed base for V; they may also denote the corresponding points on T or A. Similarly λ denotes an element of Λ; but note that $\lambda_1, \ldots, \lambda_{2n}$ is a base for Λ, rather than a set of coordinates for λ. Finally E and H denote respectively the alternating and the Hermitian version of a Riemann form on $V \times V$ with respect to Λ. By convention, a Hermitian form will be $\mathbf{C}$-linear in its first argument and $\mathbf{C}$-antilinear in its second; the reader is warned that Weil [17] adopts the opposite convention.

Since the distinction between Theorems and Lemmas is purely subjective, they share a common system of numbering; however this principle has not been extended to Corollaries.

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