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OPERATIONS OF THE NTH KIND IN K-THEORY, AND WHAT WE DON'T KNOW ABOUT RP$^{\infty}$

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I. Operations of the nth kind in K-theory

In the old days, if you wanted to solve some concrete problem in homotopy theory, you began by calculating the ordinaty cohomology groups of all the spaces involved. Then you used primary cohomology operations, such as cup-products and the Steenrod operations. If, or when, those didn't yield enough information you tried secondary ones, and then tertiary and higher ones. Of course, if the problem needed tertiary operations you didn't publish the argument in that form because it was too nasty. However, it was sometimes possible to avoid some of the nastiness by using suitable formal machinery like the Adams spectral sequence.

A little later we realised, with great pleasure, that sometimes by using a generalised cohomology theory - perhaps with primary operations - you could successfully tackle a geometrical problem which if done by ordinary cohomology would have needed operations of arbitrarily high kind. It was always conceded that the choice of the cohomology theory most useful for a particular problem might take hard work, or luck, or both. But there was a sort of democratic movement, which proclaimed that every generalised cohomology theory deserved equal rights. For example, Atiyah showed that it is technically possible to teach K-theory before ordinary cohomology. Of course all normal people still did their calculations in ordinary cohomology first, but they were made to feel that they were mere slaves of habit. The philosophy prevailed that all the apparatus of calculation, with which we are so familiar in the ordinary case, should be set up for generalised cohomology. It was conceded that some results like the Kunneth theorem might require restrictive hypotheses. On the other hand, basic things like cohomology operations seem to arise from mere category-theory, and one imagined that they would work in fair generality.

It is now time to explain that the work to be presented is joint work of David Baird and myself. David Baird had been studying classical complex K-theory localised at an odd prime p; and by any standards this is a very good cohomology theory. David arrived at a belief which I will state in the the following simplified form: in this theory you may be able to set up the apparatus of stable tertiary and higher operations, but the information yielded will be exactly zero. It is only fair to say that at first I found this suggestion both implausible and unwelcome. However, I have convinced myself that it is well founded.

To make the suggestion precise, we need machinery to estimate the scope and field of action for stable tertiary and higher operations in a given theory. In the classical case one uses homological algebra, introducing Tor and Ext over the Steenrod algebra. It is natural to try and carry this approach over to the generalised case. Given a spectrum E, you can form $E^*(E)$, the E-cohomology of the spectrum E - this is the graded algebra of stable primary operations on E-cohomology. It has to be considered as a topological algebra. Novikov took this line with success in the case of complex cobordism $E = MU$.

In the case of K-theory I think that Graeme Segal made some tentative calculations with rings of operations generated by operations Ψ^k with k prime to p. But here one suffers from a certain lack of confidence that the algebra is well related to any geometry one can do, so that one is not certain one is doing the right calculations. I believe that when the calculations seemed difficult to interpret, they were abandoned.

Later I suggested that for suitable spectra E one should consider $E_*(E)$, the E-homology of X. Under reasonable assumptions on E this behaves like the dual of the Steenrod algebra, and for any X one can make the E-homology groups $E_*(X)$ into a comodule over the coalgebra $E_*(E)$. One can form Ext of comodules over this coalgebra, and one can be sure that this homological algebra is well related to suitable geometry. (See my lectures in Springer Lecture Notes No. 99).

We are now close to a theorem. Let K be classical complex K-theory localised at an odd prime p. Let X and Y be (say) finite CW-complexes. Then we can form the reduced groups $\tilde{K}_*(X)$, $\tilde{K}_*(Y)$

and regard them as comodules over the coalgebra $K_*(K)$. Sufficient information on $K_*(K)$ has been published by Adams, Harris and Switzer. We can form $\mathrm{Ext}^{s,t}_{K_*(K)}\left(\tilde{K}_*(X), \tilde{K}_*(Y)\right)$.

Theorem. $\mathrm{Ext}^{s,t}_{K_*(K)}\left(\tilde{K}_*(X), \tilde{K}_*(Y)\right) = 0$ for $s \geq 3$.

This theorem has something to displease everybody. On the one hand, some of us are to some extent true English problem-solvers, and so we like tertiary operations, and when we are told that in this context they are useless, that is a source of pain and grief. On the other hand, some of us are to some extent true French Bourbakistes, and so we hate these inelegant higher operations and would love a theorem which dispenses us from ever considering the nasty things again. But this theorem won't do that, because it says you may still need secondary operations, and we can give examples where you do need them.

Perhaps it will be best to suggest how we should go on from here. The groups $\mathrm{Ext}^{s,t}_{K_*(K)}\left(\tilde{K}_*(X), \tilde{K}_*(Y)\right)$ should be the E_2 term of an Adams spectral sequence, but it should not converge to ordinary stable homotopy theory. However, there is a plausible candidate for the groups to which it should converge. To construct them, we start from some category C in which we can do stable homotopy theory. We then construct a category of fractions F. More precisely, F comes equipped with a functor $T : C \to F$, which has the following two properties.

(i) If $f : X \to Y$ is a morphism in C such that $K_*(f) : K_*(X) \to K_*(Y)$ is iso, then $Tf : TX \to TY$ is invertible in F.

(ii) $T : C \to F$ is universal with respect to property (i).

Heuristically, the category F would give an account of stable homotopy-theory, so far as it can be seen through the spectacles of K-theory. For example, it is plausible that an Adams spectral sequence starting from $\mathrm{Ext}^{s,t}_{K_*(K)}\left(K_*(X), K_*(Y)\right)$ should converge to the set of morphisms in F from X to Y.

By constructing F we should get a theory with all the same formal properties as stable homotopy theory, but with different coefficient groups. For example, in ordinary stable homotopy theory the group $C[S^{n+r}, S^n]$ contains a summand Z_p if $r = 2(p-1)q - 1$ with

q prime to p, $q > 0$. I presume (but it is not proved) that in the category of fractions the group $F[S^{n+r}, S^n]$ would actually be Z_p if $r = 2(p - 1)q - 1$ with q prime to p, whether q is positive or negative. The price for gaining a certain amount of periodicity is that you lose the Hurewicz theorem.

I would hope that to calculate the set of morphisms $F[X, Y]$ in this category of fractions would be quite reasonable, although we do not yet have the theorems which would enable us to do it. If so, then the category of fractions F might become useful as a computational tool. It is possible that F would retain just enough of the phenomena in C to have some interest.

One final question: should there exist also a version of unstable homotopy-theory as seen through the eyes of K-theory?

II. What we don't know about RP^{∞}

Here I would like to advertise an unsolved problem, in the hope of throwing it open to wider participation.

When we consider the iterated suspension homomorphism in unstable homotopy groups of spheres we need information about the homotopy groups of truncated real projective spaces RP^{n+r}/RP^n, and in particular about the stable homotopy groups of these spaces. Now, for r fixed the stable homotopy type of RP^{n+r}/RP^n is a periodic function of n; this allows us to speak of the stable homotopy type of RP^{n+r}/RP^n even when n is negative, interpreting it by periodicity; that is, if n is negative we interpret the stable homotopy type of RP^{n+r}/RP^n as that of RP^{n+r+2^m}/RP^{n+2^m} (but shifted down by 2^m dimensions), where m is appropriately large. This periodicity allows us to speak in certain ways which are pleasantly simple and dramatic. For example, it is more memorable to talk about dimension -1, rather than about a positive dimension congruent to -1 modulo 2^m for m appropriately large.

If we filter the spaces RP^{n+r}/RP^n by their subspaces RP^{n+s}/RP^n, and apply stable homotopy, we obtain a spectral sequence. The E^2 term of this spectral sequence has the following form:

$$E^2_{p,q} = \begin{cases} Z_2 \otimes \pi^S_q(S^0) & (p \text{ odd}) \\ \mathrm{Tor}^Z_1(Z_2, \pi^S_q(S^0)) & (p \text{ even}). \end{cases}$$

Here $\pi^S_q(S^0)$ is the q^{th} stable homotopy group of the sphere. I emphasise that this equation is intended to be equally valid for $p > 0$ and for $p \le 0$. For fixed finite q and r, $E^r_{p,q}$ is periodic in p with period 2^m for some m depending on q and r. The groups $E^\infty_{p,q}$ may be defined, but they are not periodic.

There is some evidence for the following conjecture.

Conjecture (after Mahowald). This spectral sequence converges to $\pi^S_*(S^{-1})$.

The sphere S^{-1} of stable dimension -1 appears because we have

$$RP^{-1}/RP^{-n} \simeq S^{-1} \vee (RP^{-2}/RP^{-n}).$$

This equation, of course, has to be interpreted as a statement about the stable homotopy type of RP^{2^m-1}/RP^{2^m-n} for m sufficiently large, as explained above.

It is tempting to talk about this conjecture in picturesque language, and speak as if there were a spectrum which is like the suspension spectrum of RP^∞, but has one cell in each dimension p whether p is positive, negative or zero. Just as the cohomology ring $H^*(RP^\infty; Z_2)$ is the polynomial ring $Z_2[x]$, where $x \in H^1(RP^\infty; Z_2)$, so the cohomology of this hypothetical spectrum should be the ring of finite Laurent series $Z_2[x, x^{-1}]$. It is now obvious what the cohomology of this hypothetical spectrum should be as a module over the Steenrod algebra. In fact, for $n \ge 0$ we have

$$Sq^i x^n = c(n,i)\, x^{n+i},$$

where the coefficient $c(n, i) \in Z_2$ is a periodic fraction of n; we define the coefficient $c(n, i)$ for negative values of n by periodicity and define $Sq^i x^n$ accordingly. This construction does make $Z_2[x, x^{-1}]$ into a

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module over the mod 2 Steenrod algebra; in fact, the Adem relations are satisfied, because it is sufficient to observe that they hold on x^n when n is positive. One can now make simple calculations. For example, let $Sq = \sum_{i\geq 0} Sq^i$; then $Sq\, x = x(1+x)$ and Sq is multiplicative, so $Sq\, x^{-1} = x^{-1}(1+x)^{-1} = x^{-1} + 1 + x + x^2 \ldots$, that is $Sq^i x^{-1} = x^{i-1}$.

It is necessary to point out firmly that there is no spectrum, in Boardman's category or in any other sensible category, whose cohomology is this A-module. In fact, if there were, then the generator for the 0-dimensional homology group would come from a finite subspectrum, say X. Then we could choose i so large that x^{-i} would restrict to zero in $H^{-i}(X; Z_2)$. But then $(\chi\, Sq^i)x^{-i} = 1$ would restrict to zero in $H^0(X; Z_2)$, a contradiction.

The 'hypothetical spectrum' is therefore a mythical beast. The statements we want to make do not refer to it: they are all to be interpreted in terms of finite complexes RP^{n+r}/RP^n and limits.

One line of thought which might lead one towards the conjecture stated is the consideration of

$$\mathrm{Ext}_A^{**}(Z_2[x, x^{-1}], Z_2).$$

First we consider Ext_{A_n}, where A_n is the subalgebra of A generated by $Sq^1, Sq^2, \ldots, Sq^{2^n}$. As a module over A_n, $Z_2[x, x^{-1}]$ is generated by the powers x^i with $i \equiv -1 \bmod 2^{n+1}$. We may filter $Z_2[x, x^{-1}]$ by the A_n-submodule generated by these generators x^i with $i = 2^{n+1}r - 1$, $r \leq p$. The successive subquotients have the form $A_n \otimes_{A_{n-1}} Z_2$, and their Ext groups are given by

$$\mathrm{Ext}_{A_n}^{**}(A_n \otimes_{A_{n-1}} Z_2, Z_2) \cong \mathrm{Ext}_{A_{n-1}}^{**}(Z_2, Z_2).$$

If we have to conjecture the structure of

$$\mathrm{Ext}_{A_n}^{**}(Z_2[x, x^{-1}], Z_2),$$

the simplest conjecture is

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$$\sum_{r \in Z} \mathrm{Ext}^{**}_{A_{n-1}} (Z_2, Z_2),$$

on generators of dimension $2^{n+1}r - 1$. This conjecture is true for $n = 2, 3$. We then have

$$\mathrm{Ext}^{**}_{A} (Z_2[x, x^{-1}], Z_2) = \lim_{\underset{n}{\leftarrow}} \mathrm{Ext}^{**}_{A_n} (Z_2[x, x^{-1}], Z_2);$$

if we make the simplest conjecture for the maps of the inverse system, the inverse limit would be $\mathrm{Ext}^{**}_{A} (Z_2, Z_2)$, on one generator of dimension -1.

Alternatively, we might proceed as follows. Let M be the sub-A-module of $Z_2[x, x^{-1}]$ generated by the x^i with $i \neq -1$. If M were connected, we would count its generators by computing $Z_2 \otimes_A M = 0$. That is, M is a module 'with no generators'. Similarly, $\mathrm{Tor}^A_1(Z_2, M)=0$; that is, M has 'no relations' between its 'no generators'. If M were connected it would follow that $\mathrm{Ext}^{**}_{A} (M, Z_2) = 0$; as M is not connected this does not follow, but we might still conjecture it.

Unfortunately, such statements are unlikely to lead anywhere, because even if there is an Adams spectral sequence starting from $\mathrm{Ext}^{**}_{A}(Z_2[x, x^{-1}], Z_2)$, it is likely to be very hard to prove anything useful about its convergence. We therefore turn to other evidence. In what follows, the groups $E^{\infty}_{p,q}$ are those of the spectral sequence in the conjecture.

Proposition 1. For $q = 0$, the groups $E^{\infty}_{p,0}$ are zero except for $p = -1$; $E^{\infty}_{-1,0} = Z_2$.

We can even state the differentials which lead to this state of affairs. Let us write i_n for the homology generator in dimension n (with coefficients Z or Z_2 as may be needed). Then we have

$$d_2 i_p = \eta i_{p-2} \quad \text{for } p \equiv 1 \mod 4$$
$$d_4 i_p = \nu i_{p-4} \quad \text{for } p \equiv 3 \mod 8$$
$$d_8 i_p = \sigma i_{p-8} \quad \text{for } p \equiv 7 \mod 16$$
$$d_9 i_p = \sigma\eta i_{p-9} \quad \text{for } p \equiv 15 \mod 32$$

$$d_{10}i_p = \sigma\eta^2 i_{p-10} \quad \text{for } p \equiv 31 \mod 64$$

$$d_{12}i_p = \zeta i_{p-12} \quad \text{for } p \equiv 63 \mod 128$$

$$d_{16}i_p = \rho i_{p-16} \quad \text{for } p \equiv 127 \mod 256 \text{ etc.}$$

Proposition 2. For $q = 1$, the groups $E^\infty_{p,1}$ are zero except for $p = -2$; $E^\infty_{-2,1} = Z_2$.

The group for $p = -2$ arises as follows. The complex RP^0/RP^{-3} has the form $(S^{-2} \vee S^{-1}) \cup e^0$, where the components of the attaching map for e^0 are η and 2. So in RP^0/RP^{-n}, the element $2 \in \pi^S_{-1}(S^{-1})$ compresses to RP^{-2}/RP^{-n}, and gives a generator for the group $E^\infty_{-2,1}$.

For some distance we can inspect the differentials which lead to this state of affairs. For $p \equiv 3 \mod 4$, $E^2_{p,1}$ consists of boundaries. We have

$$d_2\eta i_p = \eta^2 i_{p-2} \quad \text{for } p \equiv 0, 1 \mod 4$$

$$d_6\eta i_p = \nu^2 i_{p-6} \quad \text{for } p \equiv 2 \mod 8.$$

At this point one guesses that $d_{14}\eta i_p = \sigma^2 i_{p-14}$ for $p \equiv 6 \mod 16$, but this is wrong. We have

$$d_8\eta i_p = \bar{\nu} i_{p-8} \quad \text{for } p \equiv 6 \mod 16$$

$$d_9\eta i_p = \sigma\eta^2 i_{p-9} \quad \text{for } p \equiv 14 \mod 32.$$

It may be as well to point out the first place where we get an element of $\pi_{i-1}(S^{-1})$ with $i > 0$.

Proposition 3. For $q = 3$ and p odd, the groups $E^\infty_{p,3}$ are zero except for $p = -3$; $E^\infty_{-3,3} = Z_2$; the element $\eta \in \pi_0(S^{-1})$ compresses to RP^{-3}/RP^{-n} and gives a generator for $E^\infty_{-3,3}$.

We would be on firmer ground, of course, if we could prove that $E^\infty_{p,q} = 0$ for $p + q < -1$ - for the conjecture implies this. Applying S-duality to the finite complexes and then passing to the limit, we would like to prove something like this:

Conjecture. The group of maps in Boardman's category from the suspension spectrum of RP^∞ to the suspension spectrum of S^n is zero if $n > 0$.

This would seem to be a stable analogue of one of the following two related conjectures, which are due to Sullivan. For the first, let X be a finite simplicial complex on which Z_2 acts simplicially; let Z_2 act on S^∞ by the antipodal map.

Conjecture. The evident map, from the fixed-point set of Z_2 in X, to the function space of equivariant maps from S^∞ to X, induces an isomorphism of mod 2 cohomology.

Conjecture. If Y is a finite complex, the function-space of base-point-preserving maps from RP^∞ to Y is contractible.

The conjectures as stated seem to be inaccessible. Even if we replace them by suitable stable analogues, there is reason to think that no reformulation will be simultaneously convenient to prove and useful for the present purpose.

Finally, the Kahn-Priddy theorem is relevant to the present conjecture; it proves that for $p = -1$, $E^\infty_{-1,q}$ is zero for $q > 0$. It is just possible that comparable methods, based on the consideration of infinite loop-spaces, would be helpful in studying the present conjecture. The plan would be to take the suspension-spectrum $(RP^N/RP^{-n})/S^{-1}$, replace it by an Ω-spectrum, obtain information on the homology of the loop-spaces in the Ω-spectrum, and pass to limits.

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THE PONTRJAGIN DUAL OF A SPECTRUM

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Recall, if G is a discrete abelian group and c(G) is its character group, that is,

$$c(G) = \mathrm{Hom}(G, R/Z),$$

then Pontrjagin duality provides an isomorphism

$$H^q(X; c(G)) \approx c(H_q(X; G)).$$

The aim of this talk is to describe, without proofs, how this duality can be incorporated into generalized homology and cohomology theories and spectra.

Regarding spectra, we work in Boardman's graded homotopy category of spectra $\mathcal{S}_{h*}$ ([5]). We denote the morphisms of degree q ($f: A \to S^q B$) by

$$\{A, B\}_q.$$

If A is a spectrum and X is a CW complex or a spectrum, $A_q(X)$ and $A^q(X)$ denote the homology and cohomology of X with coefficients in A, respectively.

Let $\mathcal{G}$ be the category of discrete abelian groups. We ignore the topology on c(G) so that $c: \mathcal{G} \to \mathcal{G}$. Recall c takes exact sequences into exact sequences and direct sums into direct products. Hence for any spectrum A, $c(A_*(\ ))$ is an additive cohomology theory on the category of all CW complexes. Therefore by [2], there is a spectrum A' and a natural equivalence

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